HST.584 / 22.561 Problem Set #5 Solutions

Figures courtesy Mark Khachaturian

Marking Scheme: Question 1 – 2 points, Question 2 – 3 points, Question 3 – 3 points, Question 4 – 2 points

1) For the standard IR sequence, we can write $M_T \propto M_0(1 - 2e^{-TI/T1})$. Given that TR >> T1, we can assume that we return to equilibrium for each pulse. The easiest way to do this derivation is with a differential signal approach (if you try and use two different materials represented by T_{1a} and T_{1b}, you will obtain an expression that cannot be solved exactly). First, we determine the differential signal with respect to T_1:

$\dfrac{dM_T}{dT_1} \propto \dfrac{-2M_0 e^{-TI/T1}TI}{T_1^2}$. Now, we wish to maximize this expression with respect to TI.

We take the derivative, $\dfrac{d}{dTI}\left(\dfrac{dM_T}{dT_1}\right) \propto \dfrac{-2M_0}{T_1^2}e^{-TI/T1}(1 - \dfrac{TI}{T_1})$, set equal to 0 and solve to

find that TI = T1. Really, this has provided us with the TI that maximizes contrast, but we'll use it as a first approximation to maximizing CNR.

2-a) Here we sketch two different signal evolutions as a function of TI for two different values of T_1. The effect of taking the magnitude of the signal means we flip the negative signal portions up to same polarity as our later signal
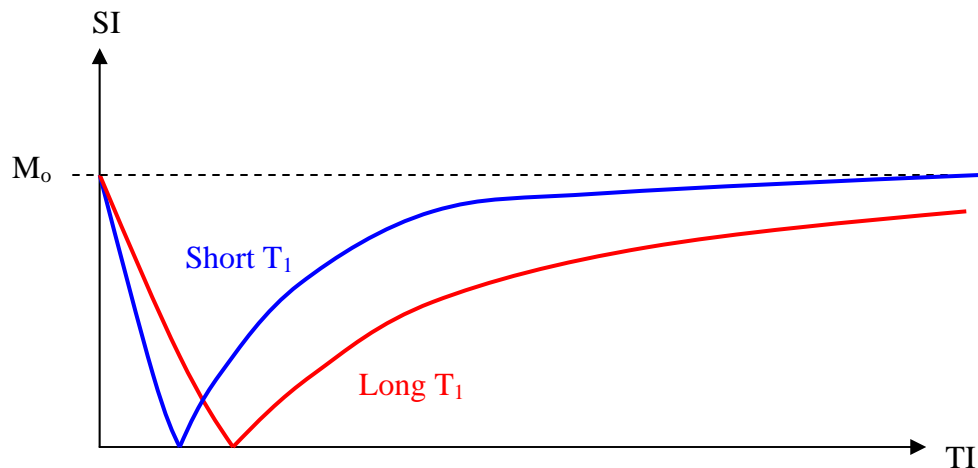
b) We wish to determine the point in the above figure where the red and blue lines cross. Using the notation $T_{1w} = 800$ ms for white matter and $T_{1g} = 1000$ ms for grey matter, we find

$$(1 - 2e^{-TI/T_{1w}}) = -(1 - 2e^{-TI/T_{1G}})$$
$$1 - e^{-TI/T_{1w}} = e^{-TI/T_{1G}}$$

Solving this numerically, we find TI = 618.0 ms. This is less than the TI we determined in the first question (TI for maximum contrast), and is definitely problematic because grey and white matter will produce the same signal (i.e. no contrast!). We have two choices to avoid this problem – we could lengthen TI (although then we have long imaging times) or shorten TI (although we will again lose CNR if we shorten it too much). In either case, we should be mindful of the zero-crossing points of our tissues – these TI would generate no signal.

c) Using TI < 0.69 for a spin echo acquisition ($M_T \alpha\ e^{-TE/T2}$) will have the net effect of increasing the overall image contrast, assuming our pathology has elevated both T_1 and T_2. This is most easily seen by picking two time points (one before and one after the cusps) and drawing T_2 decay curves for the signal available. Prior to the cusp, an elevation in T_2 shifts the signal curves away (increased contrast); after the cusp the signal decay curves shift towards each other and perhaps even cross (less contrast). The advantages of this sequence include good contrast and an ability to null the fat signal. Disadvantages include the extra acquisition time due to the use of the spin echo.

3-a)
We will track the longitudinal and transverse magnetization at each point in the sequence as if were running this sequence for the first time. I gave full credit for answers such as that because we did not explicitly ask for the steady-state solution (i.e. many repetitions of the same sequence). I will provide that derivation here as well:

Time Point	M_Z	M_T
0	M_0	0
1	0	M_0
2	$M_0 (1 - e^{-TE/(2\,T1)})$	$M_0\, e^{-TE/(2\,T2)}$
3	$-M_0 (1 - e^{-TE/(2\,T1)})$	$M_0\, e^{-TE/(2\,T2)}$
4	$M_0 (1 - 2\, e^{-TE/(2\,T1)} + e^{-TE/T1})$	$M_0\, e^{-TE/T2}$
5	$M_0 (1 - 2\, e^{-(a-TE/2)/T1} + e^{-a/T1})$	0
6	$-M_0 (1 - 2\, e^{-(a-TE/2)/T1} + e^{-a/T1})$	0
7	$M_0 + (Mz(6) - M_0)\, e^{-TI/T1}$	0
8	0	$M_0 + (Mz(6) - M_0)\, e^{-TI/T1}$
9	$M_0 (1 - e^{-TE/(2\,T1)})$	$[M_0 + (Mz(6) - M_0)\, e^{-TI/T1}]e^{-TE/(2\,T2)}$
10	$-M_0 (1 - e^{-TE/(2\,T1)})$	$[M_0 + (Mz(6) - M_0)\, e^{-TI/T1}]e^{-TE/(2\,T2)}$
11	$M_0 (1 - 2\, e^{-TE/(2\,T1)} + e^{-TE/T1})$	$[M_0 + (Mz(6) - M_0)\, e^{-TI/T1}]e^{-TE/T2}$

So:
$$S1 = M_0\, e^{-TE/T2}$$
$$S2 = [M_0 + (Mz(6) - M_0)\, e^{-TI/T1}]e^{-TE/T2}$$
$$= M_0 [1 - 2\, e^{-TI/T1} + 2\, e^{-(a+TI-TE/2)/T1} - e^{-(a+TI)/T1}]\, e^{-TE/T2}$$

Following this into the steady-state case (multiple repetitions of the same sequence):

Time Point	M_Z	M_T

| 11	$M_0 (1 - 2\,e^{-TE/(2\,T1)} + e^{-TE/T1})$	$[M_0 + (Mz(6) - M_0)\, e^{-TI/T1}]e^{-TE/T2}$
End / 0	$M_0(1 - 2e^{-(b - TI - TE/2)/T1} + e^{-(b - TI)/T1})$	0
1	0	$M_0(1 - 2e^{-(b - TI - TE/2)/T1} + e^{-(b - TI)/T1})$
4	As above	$M_0\, e^{-TE/T2}[1 - 2e^{-(b-TI-TE/2)/T1} + e^{-(b - TI)/T1}]$
All the rest	As above	As above

Now:

$S1 = M_0\, e^{-TE/T2}\,[1 - 2e^{-(b-TI-TE/2)/T1} + e^{-(b - TI)/T1}]$

$S2 = [M_0 + (Mz(6) - M_0)\, e^{-TI/T1}]e^{-TE/T2}$

$\quad = M_0\,[1 - 2\,e^{-TI/T1} + 2\,e^{-(a + TI - TE/2)/T1} - e^{-(a + TI)/T1}]\, e^{-TE/T2}$

b) Taking the ratio the two expressions will eliminate the $e^{-TE/T2}$ factor. We will then be left with an expression in terms of the measured amplitudes S1 and S2 and all the timing parameters which we have set (a, b, TI, TE). We can rearrange this expression then to isolate T1 (or more likely solve for it numerically, because that expression is going to be unpleasant.

4-a) We have the following basic picture:
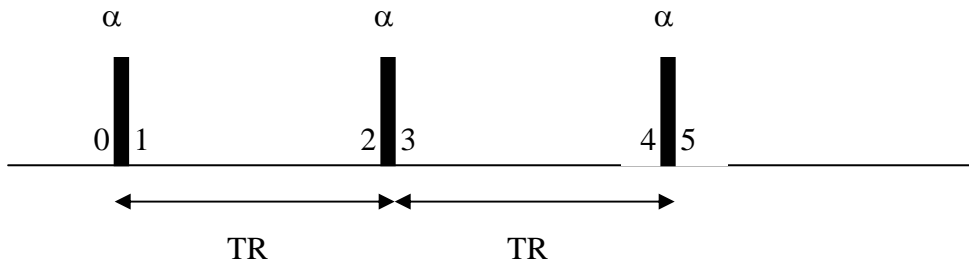
As Larry showed earlier in the term, the easiest way to derive the Ernst angles expression is to assume we have reached a steady-state (i.e. many repetitions of this sequence), which allows us to conclude that $Mz(0) = Mz(2) = Mz(4)$ etc.

Time Point	M_Z	M_T
0	$Mz(0)$	0
1	$Mz(0)\cos\alpha$	$Mz(0)\sin\alpha$
2	$M_Z(2) = M_0 + (Mz(0)\cos\alpha - M_0)e^{-TR/T1}$ $= M_Z(0)$	0

Rearranging this last expression, we get a value for the equilibrium magnetization before the pulse:

$$M_Z(0) = \frac{M_0(1 - e^{-TR/T1})}{M_0(1 - e^{-TR/T1}\cos\alpha)}$$

The signal intensity is then given by $M_Z(0)\sin\alpha$. We wish to maximize the signal with respect to α, so we take the derivative with respect to α, set it equal to 0 and solve. The answer is then $\alpha = \arccos(e^{-TR/T1})$.

b) Subbing in the values given, we get $\dfrac{M_T(\alpha)}{M_T(90)} = 2.126$. In other words, we have more than doubled our signal by using the Ernst angle with respect to a more standard 90° pulse. To compare to the spin echo sequence, we dive back into the notes and find that $M_T(SE) \propto e^{-TE/T2}(1 - e^{-TR/T1})$. Using this expression, we find $\dfrac{M_T(\alpha)}{M_T(SE)} = 3.17$ and $\dfrac{M_T(90)}{M_T(SE)} = 1.50$. The reason for this is T_2 decay. With the Ernst angle and 90° sequences, we begin acquiring signal right after the transverse magnetization is created. In the spin echo sequence, we wait time TE before acquisition, during which T_2 effects reduce the net transverse amplitude.