Lecture 6: Quantitative Aspects Of Networks III: Outline

- Network Analysis Terminology notated
- Connectivity
- Some Social Network Concepts-intuition and calculation
  - transitivity (clustering)
  - centrality
    - degree, closeness, betweenness, information, eigenvector
- prestige and acquaintance
- degree distributions
  - skew (and non-skew) distributions
  - fitting power laws to observed data
  - the normality of power laws
  - truncation
  - Structural implications and growth assumptions
- Examples of some metrics from broad Assign. # 3 "systems"
- Project Discussion

Network Analysis Terminology -notated

- node (vertex), link (edge) CM2,
- size, sparseness, metrics CM2
- degree, average degree, degree sequence DW4
- directed, simple DW4
- geodesic, path length, graph diameter DW4
- **transitivity (clustering),** connectivity, reciprocity CM6
- **centrality** (degree, closeness, betweenness, information, eigenvector) CM6
- prestige, acquaintance CM6
- ideal graphs (star, line, circle, team) CM6
- **degree distribution**, power laws, exponents, truncation, CM6

- Models (random, "small world", poisson, preferential attachment)
- constraints, hierarchy, rewiring
- community structure, cliques, homophily, assortative mixing, degree correlation coefficient
- motifs, coarse- graining
- navigation, search, epidemics and cascades
- self-similarity, scale-free, scale-rich
- dendograms, cladograms and relationship strength

Network Metrics (from lectures 2 and 4)

- n, the number of nodes
- m, the number of links
- 2m/n is the average degree <k> as the number of links on a given node, k, is the degree.
- m/[(n)(n-1)] or <k>/2(n-1)is the "sparseness" or normalized interconnection "density"
- Path length, *l*

$$l = \frac{1}{\frac{1}{2}n(n-1)}\sum_{i \geq j}d_{ij}$$

Connectivity

- **Fraction of nodes connected in a network**

- Of interest in resilience/robustness which we will cover in a later lecture and then an inverse path length definition is useful

- Can also be of interest during network formation or as links are added to a set of nodes

- Is also of interest in community determination (or decomposition) as one tries to arrive at a **disconnected** set of groups by (for example) cutting as few links as possible

- Statistical "percolation" models are often used in network formation and destruction studies and then a specific network model (for example, random) is applied to calculate connectivity.

Connectivity:
(For the Poisson random graph)

Figure by MIT OCW.

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Professor C. Magee, 2006
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Social Network Analysis

- Many structural metrics have been invented and used by Social Scientists studying social networks over the past 70+ years.

- These are well-covered in Wasserman and Faust –*Social Network Analysis* (1994) The following slides cover a ***few*** selected examples in ***one*** area *from that book*. The purpose is to give some feel for the application of such metrics which attempt to measure structural properties of direct interest for analysis

- We should also note that transitivity (clustering) and almost all other metrics discussed in this lecture were familiar to and used by social network scientists ***before*** the recent upsurge in activity over the past 7 years.

Transitivity or Clustering coefficient, C

- Measures quantitatively the degree to which nodes which each have relationships with a common node are likely to have a direct relationship.

$$C_1 = \frac{3 \, x \, number \, of \, triangles \, in \, network}{number \, of \, connected \, triples \, of \, nodes}$$

$$C_2 = \frac{1}{n} \sum_i C_i;$$

$$C_i = \frac{number \, of \, triangles \, connected \, to \, node \, i}{number \, of \, triples \, centered \, on \, i}$$

Example calculation of transitivity coefficients
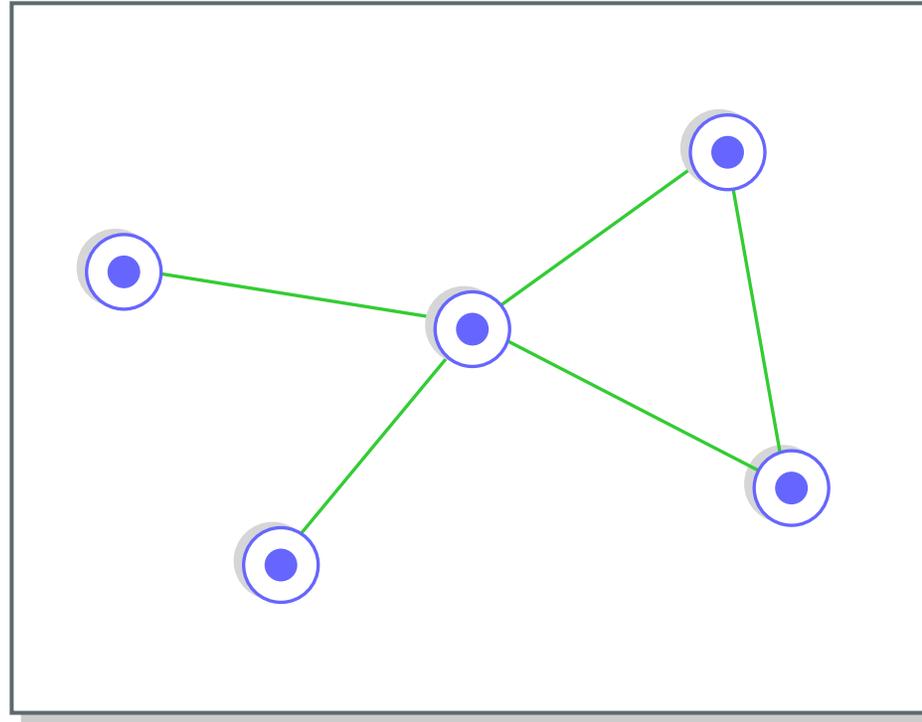
This network has one triangle and eight connected triples, and therefore has a clustering coefficient, C_1, of 3 x 1/8 = 3/8 The individual vertices have local clustering coefficients, of 1, 1, 1/6, 0 and 0, for a mean value, C_2 = 13/30.

Figure by MIT OCW.

See Newman, M. E. J. "The Structure and Function of Complex Networks." *SIAM Review* 45, no. 2 (2003): 167–256. Society for Industrial and Applied Mathematics.

Transitivity or Clustering coefficient, II

- (Almost) always > than expected from random networks thus offering some support for earlier assertions that real networks have some non-random "structure" (more later)
  - Thus, assessing transitivity is a quick check whether you have a random graph where $C = <k>/n$. Indeed the size dependence of transitivity can be useful to calculate
- Higher order clusters (groups of n related nodes) also of interest but no clean way (yet) to separate lower order and higher order tendencies
- In directed graphs, n=2 effects (the proportion of nodes that point at each other) can be of interest and is labeled **reciprocity**.

Centrality

- Numerous metrics exist in the Social Networks Literature for assessing the "centrality" of a social network.
  - Centrality metrics attempt to characterize the level of "centralization" of control or action on this network
  - One application is to assess how important **a given actor** (node) is in the network (ranking of nodes according to link information)
  - Another application is to assess **overall** how much of the control of the network is controlled by the "more important" actors
  - The relative importance of single channels/links and groups of links has also been of interest.
- We will look at a several of the social science defined metrics and explore the definitions by looking at "ideal toy graphs": Team (family or full) graphs, Circle (or line) graphs and Star graphs.

Degree Centrality

- Actor (can be individual, group or organization depending on what is being studied). The actor in the example we will use is a "Family". *Most central is the node with the most links*.

$$C_D^{'}(n_i) = \frac{\sum_j x_{ij}}{n-1}$$

- Group (all actors in network)
  - = 1 for a star graph
  - = 0 for a circle graph or "team"
  - = 1/(n-1) for line graph

$$C_D = \frac{\sum_{i=1}^{n}[C_D^{\max}(n) - C_D(n_i)]}{[(n-1)(n-2)]}$$

Padgett's Florentine Families: 15th Century Marriage Relations
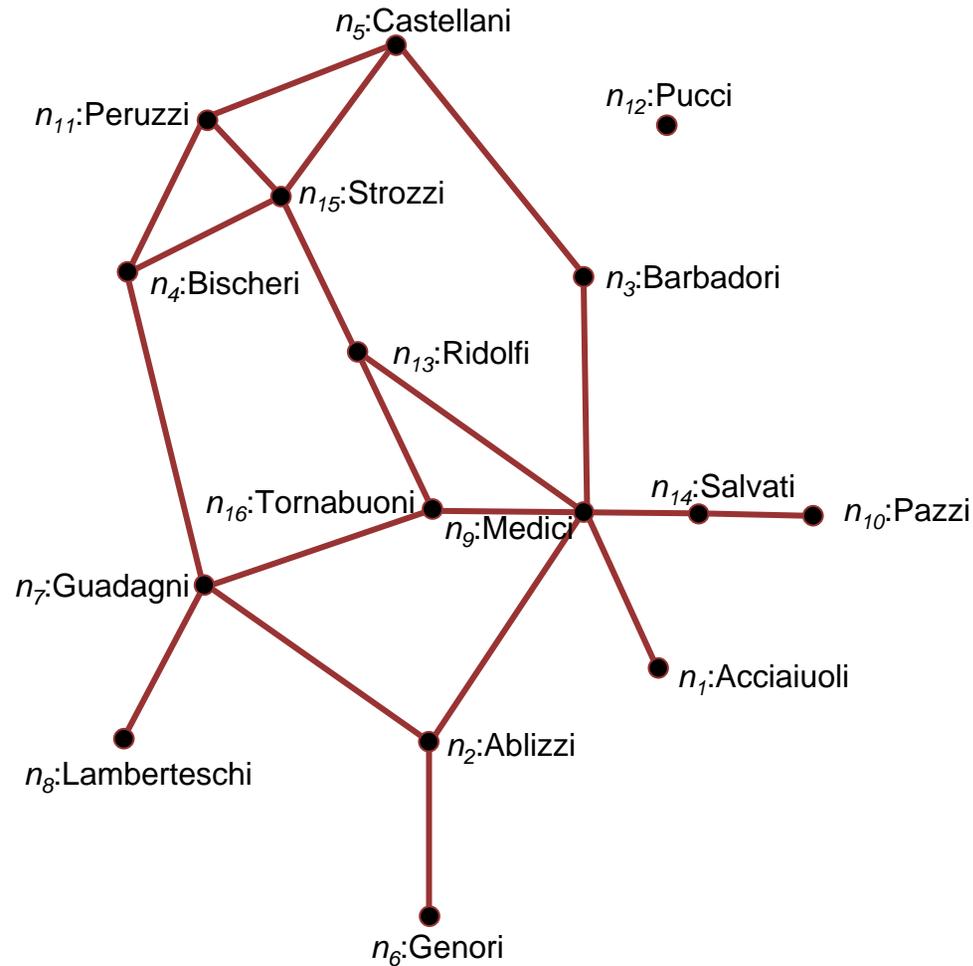
Florentine Families Centrality Metrics I: Degree

	$C'_D(n_i)$	$C'_C(n_i)$	$C'_B(n_i)$	$C'_I(n_i)$
Acciaiuoli	0.071			
Ablizzi	0.214			
Barbadori	0.143			
Bischeri	0.214			
Castellani	0.214			
Genori	0.071			
Guadagni	0.286			
Lamberteschi	0.071			
Medici	0.429			
Pazzi	0.071			
Peruzzi	0.214			
Pucci	---			
Ridolfi	0.214			
Salvati	0.143			
Strozzi	0.286			
Tornabuoni	0.214			
Centralization	**0.257**			

Closeness Centrality

- Actor
  - *Closest is shortest* (geodesic) *distance* from other nodes =1 for max closeness and 0 for min

$$C_C^{'}(n_i) = \frac{n-1}{\sum_{j=1}^{n} d(n_i, n_j)}$$

- Group
  - = 0 for circle graph or full network
  - = 1 for star graph
  - 0.277 for line (7 nodes)
  - can estimate several ways including dispersion

$$C_C = \frac{\sum_{i=1}^{n}[C_C^{i\,\max} - C_C^{'}(n_i)]}{(n-2)(n-1)/(2n-3)}$$

Florentine Families Centrality Metrics II Closeness

	$C'_D(n_i)$	$C'_C(n_i)$	$C'_B(n_i)$	$C'_I(n_i)$
Acciaiuoli	0.071	0.368		
Ablizzi	0.214	0.483		
Barbadori	0.143	0.438		
Bischeri	0.214	0.400		
Castellani	0.214	0.389		
Genori	0.071	0.333		
Guadagni	0.286	0.467		
Lamberteschi	0.071	0.326		
Medici	0.429	0.560		
Pazzi	0.071	0.286		
Peruzzi	0.214	0.368		
Pucci	---	---		
Ridolfi	0.214	0.500		
Salvati	0.143	0.389		
Strozzi	0.286	0.438		
Tornabuoni	0.214	0.483		
Centralization	**0.257**	**0.322**		

Betweeness Centrality I

- Actor
  - Power or influence comes from being an intermediary
  - z is the number of geodesics between two points

$$C_B^{'}(n_i) = \frac{\sum_{j<k} z_{jk}(n_i)/z_{jk}}{[(n-1)(n-2)/2]}$$

- Group
  - =1 for star graph
  - =0 for circle
  - =0.311 for 7 node line
  - Tree Hierarchy = xx

$$C_B = \frac{\sum_{i=1}^{n}[C_B^{'max}(n) - C_B^{'}(n_i)]}{n-1}$$

Definition of Hierarchy

- Not covered in ESD Terms and Definitions but..
- Hierarchy: A description of a group of elements (system?) where each element is graded or ranked and then arranged in a structure that separates elements according to rank which each descending rank being in some way subordinate to the next higher rank (this leads to a level number or node depth). Although hierarchy often describes power or authority relationships, it is also used in describing levels of abstraction and other system features. Closed and Open hierarchies have also been distinguished.
- Hierarchies can take on a variety of structures ranging from Pure layers to pure trees.

Florentine Families Centrality Metrics: Betweeness

	$C'_D(n_i)$	$C'_C(n_i)$	$C'_B(n_i)$	$C'_I(n_i)$
Acciaiuoli	0.071	0.368	0.000	
Ablizzi	0.214	0.483	0.212	
Barbadori	0.143	0.438	0.093	
Bischeri	0.214	0.400	0.104	
Castellani	0.214	0.389	0.055	
Genori	0.071	0.333	0.000	
Guadagni	0.286	0.467	0.255	
Lamberteschi	0.071	0.326	0.000	
Medici	0.429	0.560	0.522	
Pazzi	0.071	0.286	0.000	
Peruzzi	0.214	0.368	0.022	
Pucci	---	---	---	
Ridolfi	0.214	0.500	0.114	
Salvati	0.143	0.389	0.143	
Strozzi	0.286	0.438	0.103	
Tornabuoni	0.214	0.483	0.092	
Centralization	**0.257**	**0.322**	**0.437**	

Padgett's Florentine Families: 15th Century Marriage Relations

n_5:Castellani

n_{12}:Pucci

n_{11}:Peruzzi

n_{15}:Strozzi

n_4:Bischeri

n_3:Barbadori

n_{13}:Ridolfi

n_{16}:Tornabuoni

n_{14}:Salvati

n_9:Medici

n_{10}:Pazzi

n_7:Guadagni

n_1:Acciaiuoli

n_8:Lamberteschi

n_2:Ablizzi

n_6:Genori
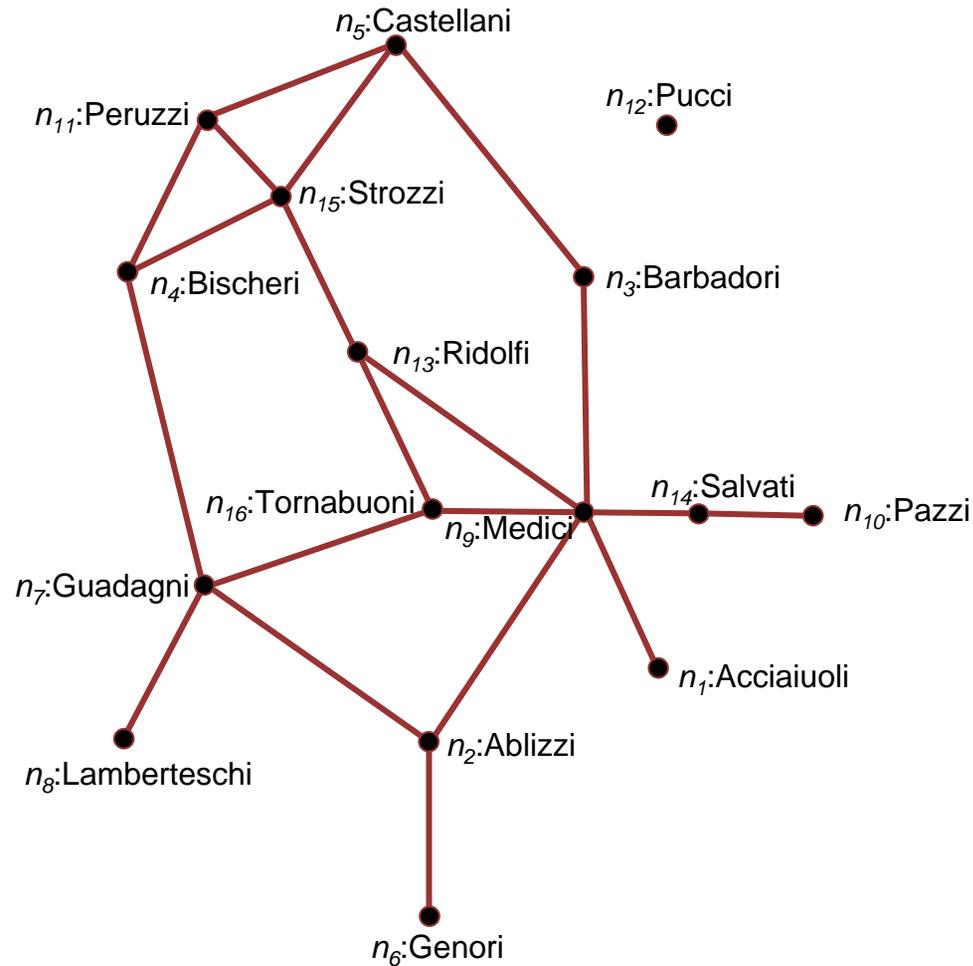
Betweeness Centrality II

- Actor
  - Power or influence comes from being an intermediary
  - z is the number of geodesics between two points
- Group
  - =1 for star graph
  - =0 for circle
  - =0.311 for 7 node line

- **_Betweeness Centrality_** has been most applied of the centrality metrics in Social Network Analysis (1994)

$$C_B'(n_i) = \frac{\sum_{j<k} z_{jk}(n_i)/z_{jk}}{[(n-1)(n-2)/2]}$$

$$C_B = \frac{\sum_{i=1}^{n}[C_B'^{\max}(n) - C_B'(n_i)]}{n-1}$$

Information Centrality

- Actor
    - Estimates the information value of the connections
    - shorter distances are better but are not the only paths used
    - T is the trace, R a row sum and c an element in a matrix constructed from the sociomatrix with information content
    - Actor indices are *proportions of total "information" flow controlled by a single actor* and sums to 1 in network
- No group index (1997)

$$C_I(n_i) = \frac{1}{c_{ii} + (T - 2R)/n}$$

$$C_I'(n_i) = \frac{C_I(n_i)}{\sum_i C_I(n_i)}$$

Florentine Families Centrality Metrics

	$C'_D(n_i)$	$C'_C(n_i)$	$C'_B(n_i)$	$C'_I(n_i)$
Acciaiuoli	0.071	0.368	0.000	0.049
Ablizzi	0.214	0.483	0.212	0.074
Barbadori	0.143	0.438	0.093	0.068
Bischeri	0.214	0.400	0.104	0.074
Castellani	0.214	0.389	0.055	0.070
Genori	0.071	0.333	0.000	0.043
Guadagni	0.286	0.467	0.255	0.081
Lamberteschi	0.071	0.326	0.000	0.043
Medici	0.429	0.560	0.522	0.095
Pazzi	0.071	0.286	0.000	0.033
Peruzzi	0.214	0.368	0.022	0.069
Pucci	---	---	---	---
Ridolfi	0.214	0.500	0.114	0.080
Salvati	0.143	0.389	0.143	0.050
Strozzi	0.286	0.438	0.103	0.070
Tornabuoni	0.214	0.483	0.092	0.080
Centralization	**0.257**	**0.322**	**0.437**	**---**

Eigenvector Centrality-UCINET

- UCINET-help, help topics, index (on toolbar), eigenvector centrality
- Given an adjacency matrix A, the centrality of vertex i (denoted ci), is given by ci =aSAijcj where a is a parameter. The centrality of each vertex is therefore determined by the centrality of the vertices it is connected to. The parameter á is required to give the equations a non-trivial solution and is therefore the reciprocal of an eigenvalue. It follows that the centralities will be the elements of the corresponding eigenvector. The normalized eigenvector centrality is the scaled eigenvector centrality divided by the maximum difference possible expressed as **a percentage**.

- For a given binary network with vertices v1....vn and maximum eigenvector centrality cmax, the network eigenvector centralization measure is S(cmax - c(vi)) divided by the maximum value possible, where c(vi) is the eigenvector centrality of vertex vi.

- This routine calculates these measures and some descriptive statistics based on these measures. This routine **only handles symmetric data** and in these circumstances the eigenvalues provide a measure of the accuracy of the centrality measure. To help interpretation the routine calculates all positive eigenvalues but only gives the eigenvector corresponding to the largest eigenvalue.

Eigenvector Centrality (from Newman and Brin and Page)

- Each node has a weight x_i that is defined to be proportional to the weights of all nodes that point to the node (i)

- And
$$x_i = \lambda^{-1} \sum_j A_{ij} x_j$$

- And then $Ax = \lambda x$
- Thus the weights are an eigenvector of the adjacency matrix (A) with eigenvalue λ

Florentine Families Centrality Metrics (with Eigenvector Centrality)

	$C'_D(n_i)$	$C'_C(n_i)$	$C'_B(n_i)$	$C'_I(n_i)$	
Acciaiuoli	0.071	0.368	0.000	0.049	.19
Ablizzi	0.214	0.483	0.212	0.074	.35
Barbadori	0.143	0.438	0.093	0.068	.30
Bischeri	0.214	0.400	0.104	0.074	.40
Castellani	0.214	0.389	0.055	0.070	.37
Genori	0.071	0.333	0.000	0.043	.11
Guadagni	0.286	0.467	0.255	0.081	.41
Lamberteschi	0.071	0.326	0.000	0.043	.12
Medici	0.429	0.560	0.522	0.095	.61
Pazzi	0.071	0.286	0.000	0.033	.06
Peruzzi	0.214	0.368	0.022	0.069	.39
Pucci	---	---	---	---	0
Ridolfi	0.214	0.500	0.114	0.080	.48
Salvati	0.143	0.389	0.143	0.050	.20
Strozzi	0.286	0.438	0.103	0.070	.50
Tornabuoni	0.214	0.483	0.092	0.080	.46
Centralization	**0.257**	**0.322**	**0.437**	**---**	.43

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Centrality II

- Numerous metrics exist in the Social Networks Literature for assessing the "centrality" of a social network.
    - Centrality metrics attempt to characterize the level of "centralization" of control or action on this network
    - One application is to assess how important a given actor (node) is in the network
    - Another application is to assess overall how much of the control of the network is controlled by the "more important" actors
    - The relative importance of single channels/links and groups of links has also been of interest.
- Centrality utility:
    - The calculation methods have been applied in search, navigation and community structure models but otherwise the "Network Science" Community does not utilize these measures. CM bias is that they are probably useful in social and other networks.
    - Hidden Hierarchy, robustness –communication and other meanings are all dependent on effects such as those defined and some of these measures (***betweenness and eigenvector centrality***) ***deserve more attention in modern network analysis***.

Related newer Centrality-like metrics

- Jon Kleinberg (Computer Science at Cornell) has done much of the leading work in search and navigation (more later).

- In some of his earliest work on this topic (1997-1999), he "invented" some useful new metrics for looking at important nodes (particularly on directed networks and probably most useful in the domain he was interested in-- the www)

- He looked for ways to find related sets of "Authorities" and "Hubs" and differentiated these from single "high in-degree nodes"

Hubs   Authorities

Unrelated page of large in-degree

A Densely Linked Set of Hubs and Authorities
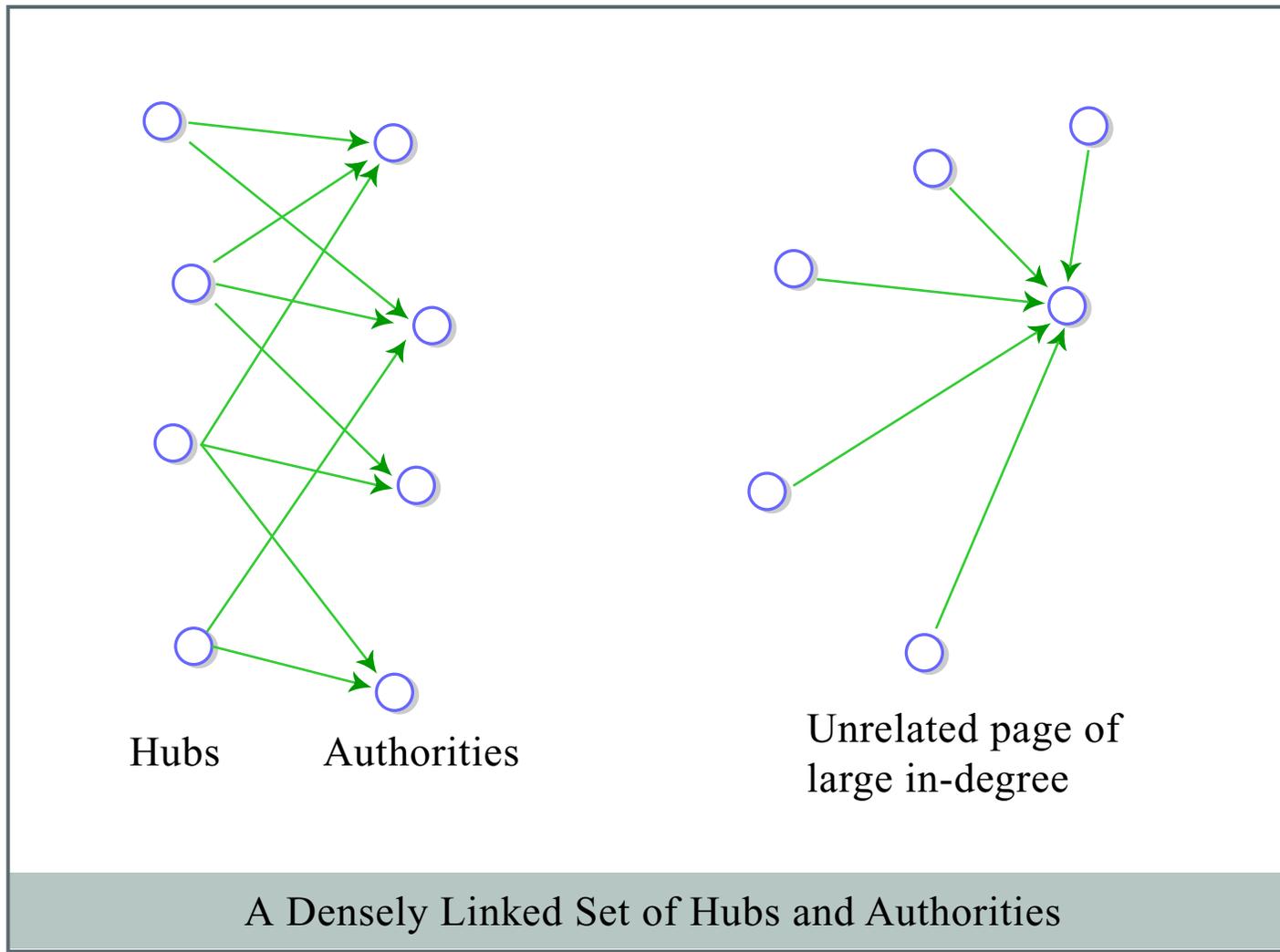
Figure by MIT OCW.

See Kleinberg, Jon. M. "Authoriative Sources in a Hyperlinked Environment" Journal of the ACM 46, no. 5 (1999): 604-632.

Prestige and Acquaintance Calculation

Authority: not only referred to by many nodes, but also by many Hubs. (measurement: *prestige*)

Hub: not only refers to many nodes, but also to many Authorities. (measurement: *acquaintance*)

A : adjacency matrix
x_i : prestige (of node i)
y_i : acquaintance

$$Ax = \lambda y$$
$$A^T y = \mu x$$

Solve for x and y.

These metrics are proving useful in directed citation networks (Mo-Han Hsieh thesis work on Internet Standards)

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  - prestige and acquaintance
- **degree distributions**
  - skew (and non-skew) distributions
  - fitting power laws to observed data
  - the normality of power laws
  - truncation
  - Structural implications and growth assumptions
- Examples of some metrics from broad Assign. # 3 "systems"
- Project Discussion

Degree Distributions

- Define p_k as the fraction of nodes in a network with degree k. This is equivalent to the probability of randomly picking a node of degree k
- A plot of p_k can be formed by making a histogram of the degrees of the nodes. **This is the degree distribution of the network**.
- Histograms
  - Normal (and nearly so)
  - Skewed (and heavily skewed)
- Suggest some normal or nearly normal distributions..

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Two nearly normal distributions
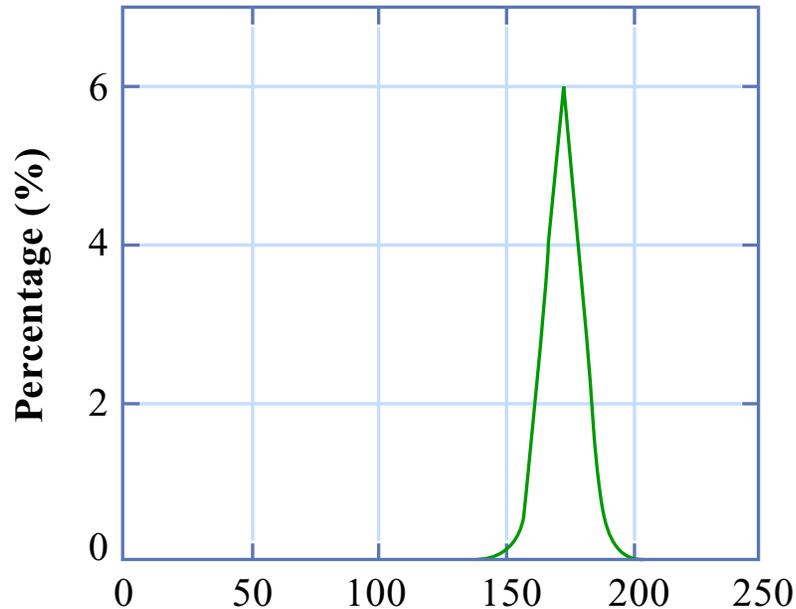
Heights of Males

Histogram of heights in centimeters of American males. Data from the National Health Examination Survey, 1959-1962 (US Department of Health and Human Services).
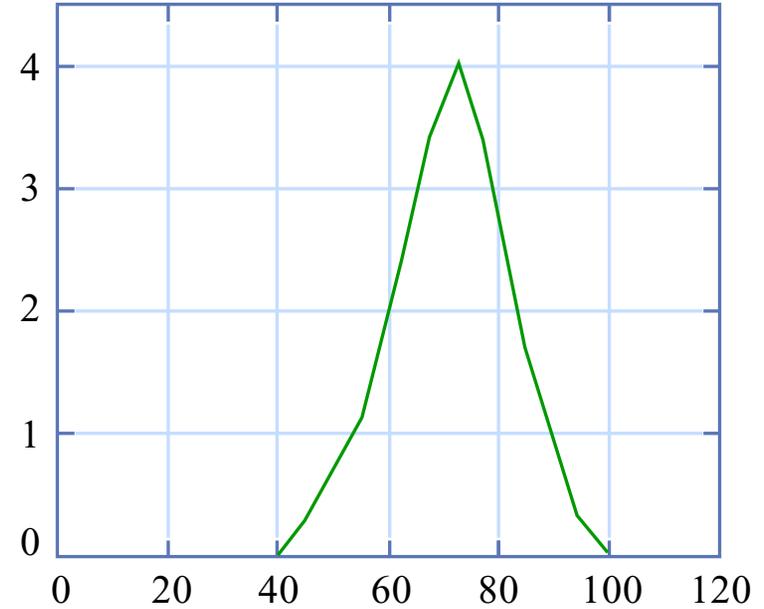
Speeds of Cars

Histogram of speeds in miles per hour of cars on UK motorways. Data from Transport Statistics 2003 (UK Department for Transport).

Figure by MIT OCW.

See Newman, M. E. J. cond-mat/0412004v2

A heavily skewed distribution

Figure by MIT OCW.

See Newman, M. E. J. cond-mat/0412004v2

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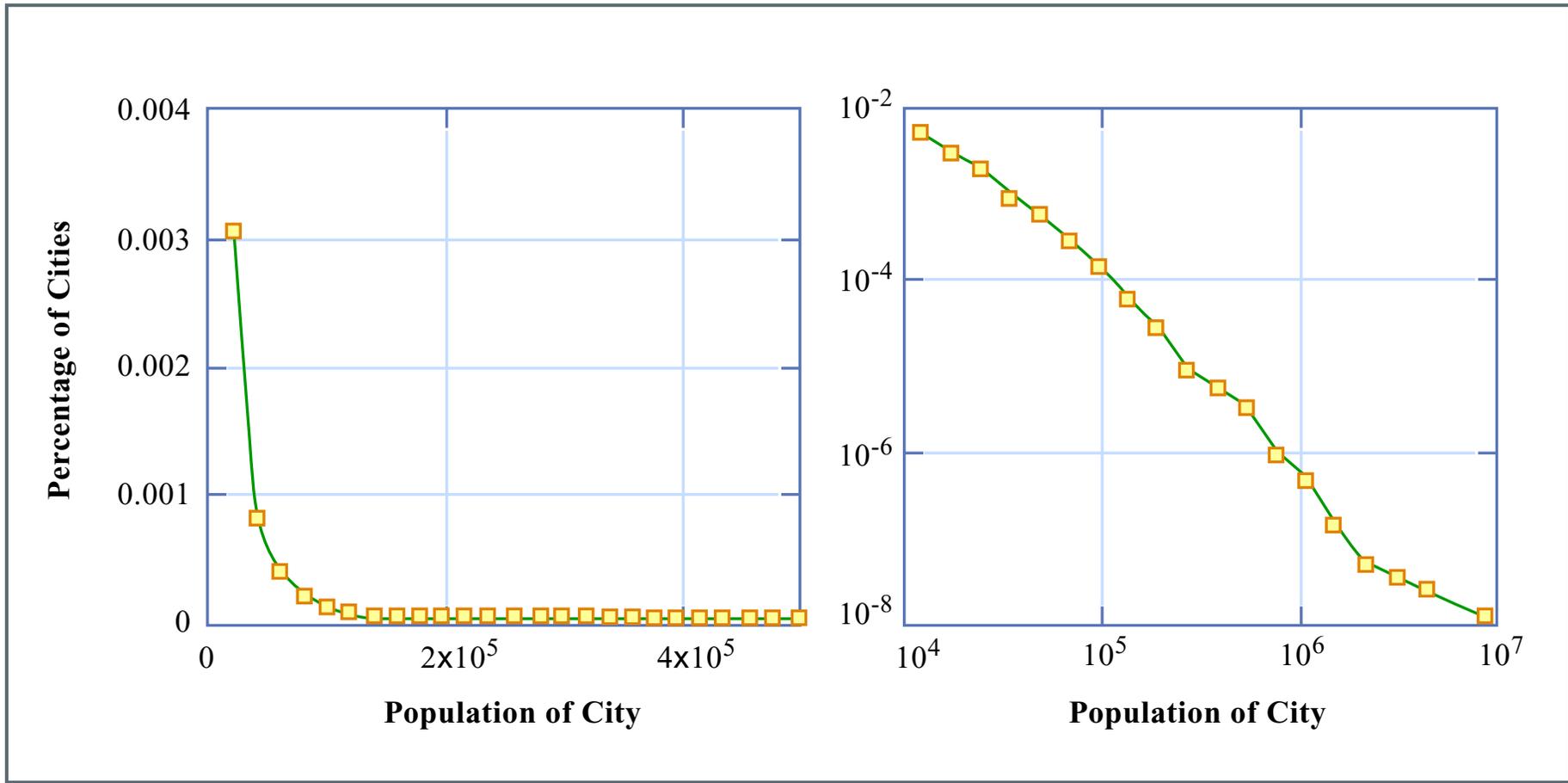
Degree Distributions II

- Define p_k as the fraction of nodes in a network with degree k. This is equivalent to the probability of randomly picking a node of degree k
- A plot of p_k can be formed by making a histogram of the degrees of the nodes. **This is the degree distribution of the network**.
- Histograms
  - Normal (and nearly so)
  - Skewed (and heavily skewed)
- Reasons for normal vs. skewed?

- Power law (skewed) $$p_k \sim k^{-\alpha}$$

- Plot ln p_k vs. ln k, slope = α
  Why might cumulative plot be superior?

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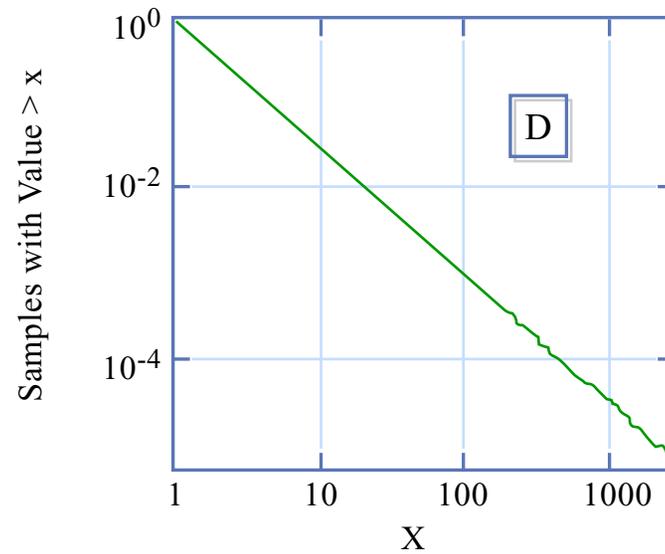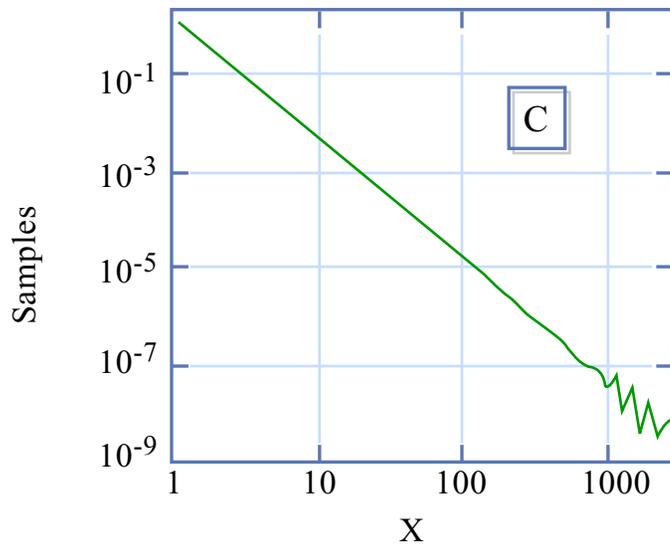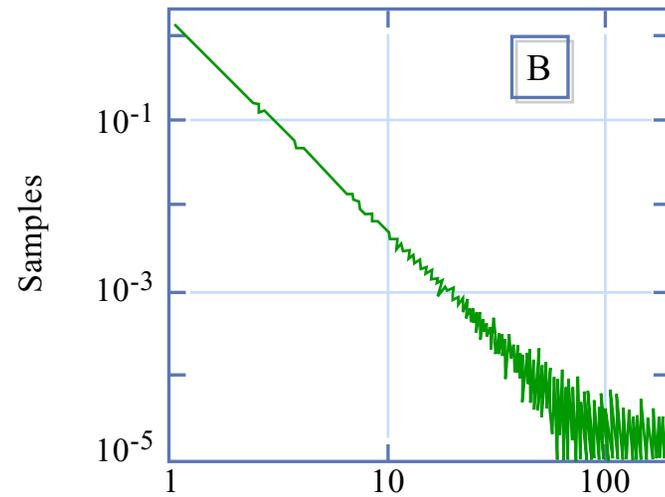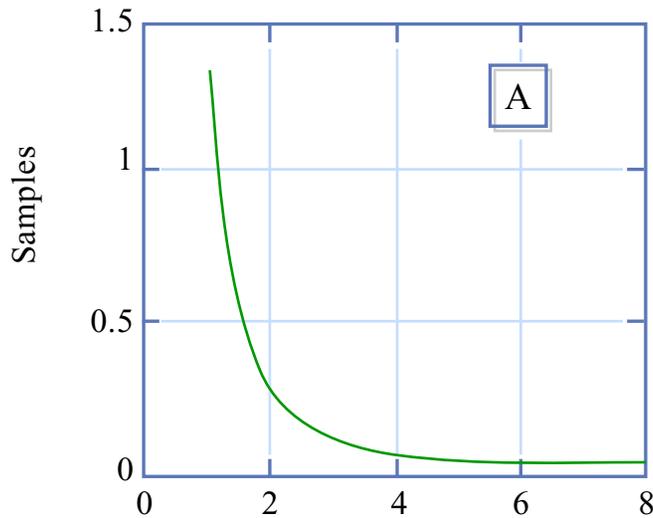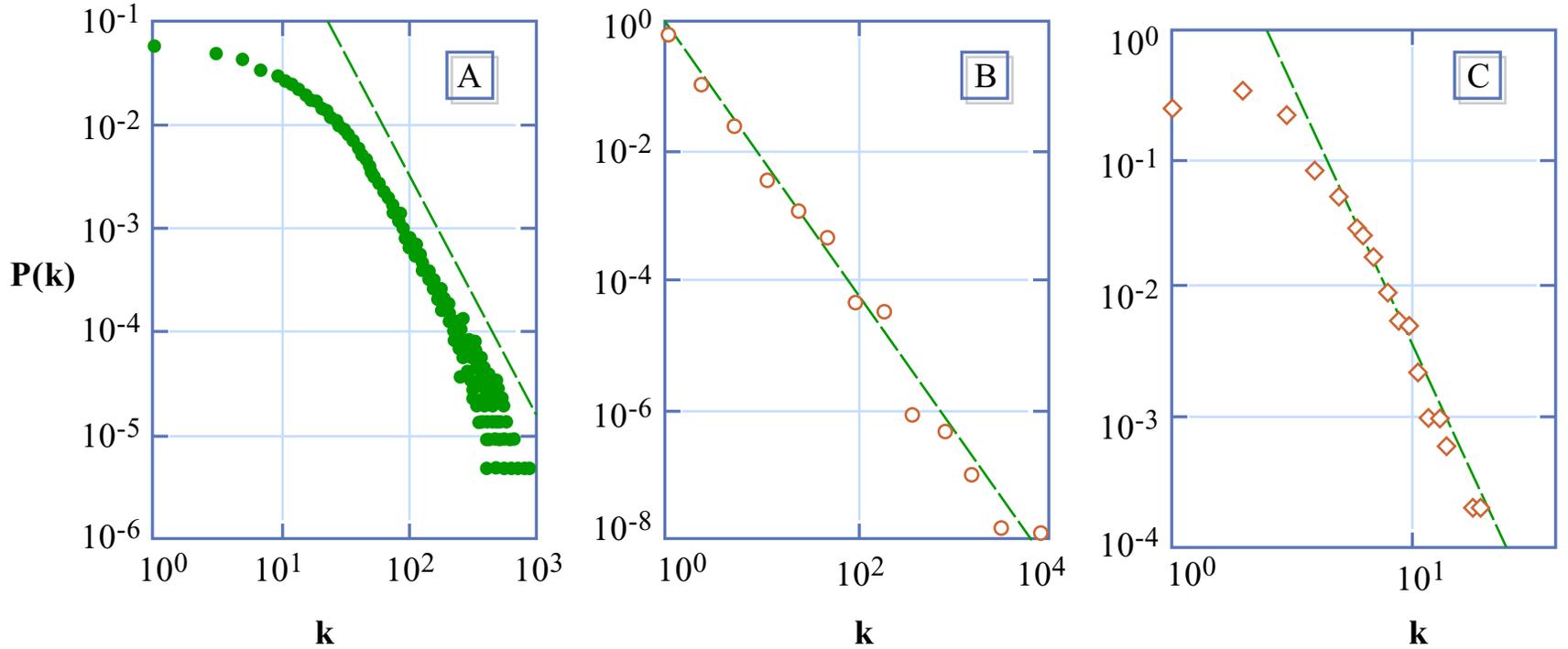
Figure by MIT OCW.

Barabasi and Albert(1999) A is actor collaboration, B is www
and C is the Western Power Grid (incorrectly identified as power law)
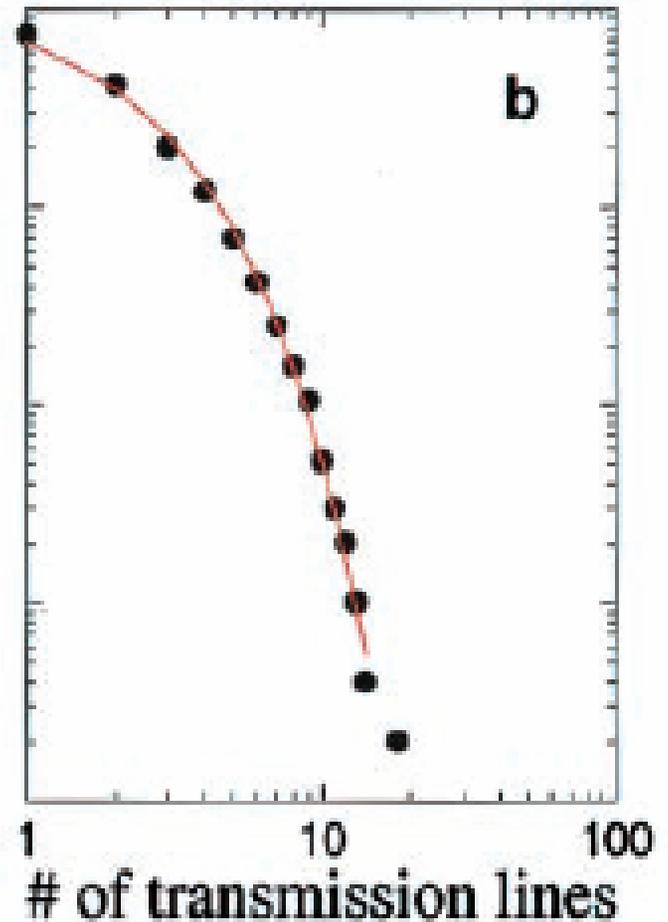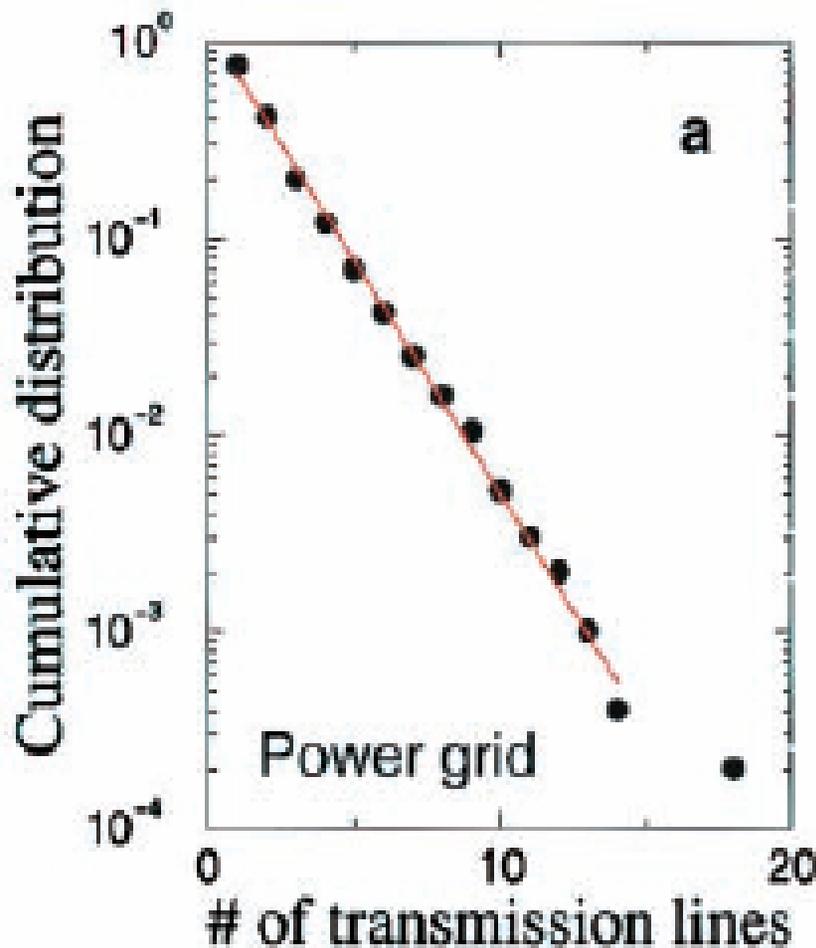
Courtesy of National Academy of Sciences, U. S. A. Used with permission.
Source: Amaral, L. A. N., A. Scala, M. Bertelemy, and H. E. Stanley. "Classes of Small World Networks." *Proc Natl Acad Sci* 97 (2000): 11149-52. (c) National Academy of Sciences, U.S.A.

From "Classes of Small World Networks," showing the same power grid data clearly **not** a power law by use of (appropriate) cumulative distribution (chart b)

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Comparison of Models with Structural Metrics : Degree distribution

- Does the existence of a power law for degree distributions for networks indicate existence of a specific mechanism for formation?

  - *No*, power laws are consistent with a wide variety of mechanisms for network formation (Newman, "Power laws, Pareto distributions and Zipf's law"2004/5)

- Does the existence of power laws for degree distributions for networks indicate the existence of a certain kind of structure for the network?

  - *No*, power laws are consistent with a wide variety of networks having various structures and some without central hubs (Li et al)

  - Moreover, power laws are *the equivalent of normal distributions at high variation* (Samorodnitsky and Taqqu)

Central Limit Theorem

The mean of a sequence of *n* iid random variables with

- Finite μ (and variance)

- $E\left(\left|x_i - E(x_i)\right|^{2+\delta}\right) < \infty \quad \delta > 0$

approximates a normal distribution in the limit of a large *n*.

From Systems engineering lecture by Dan Frey

Marginal and Markov process defined

- Marginal probability- In a multivariate distribution, the probability of one variable, or function of several of these variables, taking a specific value (or falling in a range)
  - Metric: α An **outer measure** on a product space, by restriction to one of the two factors: if μ is an outer measure on X x Y , the marginal probability is a measure that satisfies $\alpha(A) = \mu(A \, x \, Y)$

- Markov chain or process. A sequence of events, usually called states, the probability of each of which is dependent only the event immediately preceding it.

Power laws are ubiquitous

More normal than Normal

Low variability

High variability

Gaussian

Exponential

Power law

Central Limit Theorem (CLT)

Marginalization (Markov property)

CLT
Marginalization
Maximization
Mixtures

From seminar by John Doyle at GT in Nov. 2004

Comparison of Models with Structural Metrics : Degree distribution

- Does the existence of a power law for degree distributions for networks indicate existence of a specific mechanism for formation?
  - *No*, power laws are consistent with a wide variety of mechanisms for network formation (Newman, "Power laws, Pareto distributions and Zipf's law"2004/5)
- Does the existence of power laws for degree distributions for networks indicate the existence of a certain kind of structure for the network?
  - *No*, power laws are consistent with a wide variety of networks having various structures and some without central hubs (Li et al)
  - Moreover, power laws are the equivalent of normal distributions at high variation (Samorodnitsky and Taqqu)
- **Power laws are *very* useful for representation and manipulation of data but are *not* strongly indicative of structure or behavior (despite what you may read)**

Degree Distributions III

- Define p_k as the fraction of nodes in a network with degree k. This is equivalent to the probability of randomly picking a node of degree k
- A plot of p_k can be formed by making a histogram of the degrees of the vertices. **This is the degree distribution of the network**. Some distributions
- Random Graph- binomial

 (poisson at large n)

$$p_k \cong \frac{<k>^k \ e^{-<k>}}{k!}$$

- Exponential

$$p_k \approx e^{\frac{-k}{\kappa}}$$

- Power Law

$$p_k \sim k^{-\alpha}$$

- lognormal

$$p(\ln k) \approx \exp(-\frac{(\ln k - \mu)^2}{2\sigma^2})$$

Degree Distributions IV

- Other Distributions
  - Power law with exponential cutoff is "common"
  - For bipartite graphs, there are two degree distributions, one for each type of node (multipartite one for each type of node)
  - For directed graphs, each node has an in-degree and an out-degree and the degree distribution becomes a function of two variables (j and k for in and out degrees). Since in and out degrees can be strongly correlated, the joint distribution also contains information about the network.
- Maximum Degree (Power Law)

$$k_{max} \sim n^{1/(\alpha-1)}$$

Network Metrics (from lectures 2, 4 and now lecture 6)

- n, the number of nodes
- m, the number of links
- 2m/n is the average degree <k> as the number of links on a given node, k, is the degree.
- m/[(n)(n-1)] or <k>/(n-1)is the "sparseness" or normalized interconnection "density"
- Path length, *l*

$$l = \frac{1}{\frac{1}{2}n(n-1)} \sum_{i \geq j} d_{ij}$$

- Connectivity
- Clustering (2 definitions)
- Centrality (5 definitions + prestige and acquaintance)
- Degree Distribution
- Compare some systems (see handout for assignment # 3)

Networks structural characteristics: Preliminary summary of results

- Most measures-even simple ones- show that real systems (represented as networks) have "structure" (linking regularities beyond random).

- Real system architectures will not be describable by a single structural metric or feature. One must consider, size, sparseness, degree distribution, transitivity (and probably centrality and others) *simultaneously* in order to *begin* to understand a specific complex system and its similarities/differences from other complex systems.

- Although there are numerous metrics available, these are not necessarily (or even likely) to be the simplest or best to describe the systems we are interested in compactly.

- *However*, invention of new characteristics without fully understanding and exploring existing metrics is most likely to introduce unnecessary confusion rather than enlightenment (the 2 transitivity coefficients example)

References for Lecture 6

- **Overall key references**
  - Wasserman and Faust, *Social Network Analysis: Methods and Applications,* Cambridge University Press (1994)
  - M. E. J. Newman, "The structure and function of complex networks" *SIAM Review* vol. **45**, 167-256 (2003) (**posted** in assignment section)
  - J. Scott, *Social Network Analysis: A Handbook* Sage Publications (2000)
- For Centrality related
  - W & F (above) plus UCINET help and Hanneman book
  - Jon. M. Kleinberg "Authoriative Sources in a Hyperlinked Environment" Journal of the ACM, Vol. **46,** no. 5, 1999,pp 604-632
- For Power Laws
  - M. E. J. Newman, "Power Laws, Pareto Distrubutions and Zipf's law, cond-mat/0412004v2 (**will post** with lecture 6)
  - Samorodnitsky,G. and Taqqu, S., *Stable Non-Gaussian Random Processes: Stochastic Processes with Infinite Variance*, Chapman and Hall, London, (1994)
  - A Barabasi and R. Albert, "The Emergence of Scaling Laws in Random Networks", *Science* **286**, pp 509-512 (1999)
  - Amaral, L. A. N., Scala, A., Bertelemy, M. and Stanley, H. E. "Classes of Small World Networks", Proc. Nat. Acad. Sci. **97**, 11149-52 (2000)