6.890	Lecture 14	Oct. 21, 2014

Exponential Time Hypothesis (ETH):

there is no $2^{o(n)}$-time algorithm for 3SAT
\# variables [Impagliazzo & Paturi – CCC 1999]

- current best algorithm is 1.30704^n [Hertli 2011]

\# clauses

$\Leftrightarrow$ there is no $2^{o(m)}$-time algorithm for 3SAT
[Sparsification Lemma – Impagliazzo, Paturi, Zane – JCSS 2001]
- cf. $m = O(n^3)$
- dense formula $\to O(2^{\varepsilon n})$ sparse formulas $\forall \varepsilon$

Strong ETH: no $(2-\varepsilon)^n$-time alg. for CNF-SAT
(i.e. constant for k-SAT $\to 2$ as $k \to \infty$) [I&P]

(following lecture notes by Dániel Marx)

3-coloring:
- recall NP-hardness reduction from 3SAT [L9]
  [Garey, Johnson, Stockmeyer – TCS 1976]
  - n variables & m clauses
    $\to O(n+m)$ vertices & edges
- ETH $\Rightarrow$ no $2^{o(n)}$-time algorithm for 3-coloring graph where $|V|$ & $|E| = O(n)$

Size blowup of NP reduction: $|x| = n \xrightarrow{f} |x'| = b(n)$

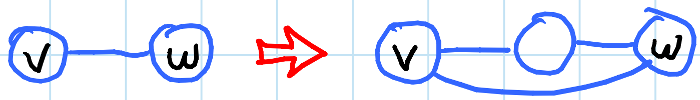
- $T(n)$ alg. for $B \Rightarrow T(b(n))$ alg. for A
- no $2^{o(n)}$ for $A \Rightarrow$ no $2^{o(b^{-1}(n))}$ for B
- b linear $\Rightarrow$ preserve "no $2^{o(n)}$-time alg."

Vertex Cover: ETH $\Rightarrow$ no $2^{o(n)}$-time algorithm for $|V|$ & $|E| = O(n)$

- e.g. L7 / Lichtenstein 1982 reduction has linear blowup

Dominating Set: ditto

- e.g. L10/Papadimitriou & Yannakakis 1991 reduction from Vertex Cover



Hamiltonicity: ditto

- e.g. L7 / Lichtenstein 1982 reduction or L8/ Plesník 1979 reduction $\Rightarrow$ max. deg. 3 has linear blowup
- not planar versions: maybe $\Theta(n^2)$ crossovers

Independent set: ditto

- e.g. L10/Papadimitriou & Yannakakis 1991 reduction from 3SAT-3
  - need -3 to avoid quadratic # edges

<u>Clique</u>: ETH $\Rightarrow$ no $2^{o(|V|)}$-time algorithm $(|E| = \Theta(|V|^2))$

<u>Planar 3SAT</u>: L7 / Lichtenstein 1982 reduction has quadratic blowup

- ETH $\Rightarrow$ no $2^{o(\sqrt{n})}$ & no $2^{o(\sqrt{m})}$-time algorithm
  - n ↳ #vars.
  - m ↳ #clauses

<u>Planar 3-coloring</u>, <u>Vertex Cover</u>, <u>Dominating Set</u>, <u>Hamiltonicity</u>, <u>Independent Set</u>: (NOT Clique)

- ETH $\Rightarrow$ no $2^{o(\sqrt{n})}$-time algorithm for planar graphs with n vertices

(above reductions) $\Rightarrow O(n)$ edges

[Cai & Juedes 2001]

<u>Parameterized consequences</u>:

- no $2^{o(k)} \cdot n^{O(1)}$ algorithm for (k-) Vertex Cover, k-Path (Longest Path), Dominating Set, Independent Set, Clique

  not surprising — not even FPT if ETH holds

- no $2^{o(\sqrt{k})} n^{O(1)}$ algorithm for Planar (no 3-coloring) Vertex Cover, Longest Path, Dom. Set, Ind. Set
  - $2^{O(\sqrt{k})} n^{O(1)}$ algorithms known

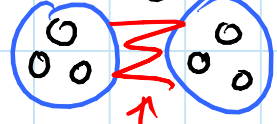
    [Alber, Bodlaender, Fernau, Kloks, Niedermeier 2002; Demaine, Fomin, Hajiaghayi, Thilikos — JACM 2005]

|V| or |E| (here n vs. n^2 not important)

Stronger: ETH $\Rightarrow$ no $f(k)\, n^{o(k)}$-time algorithm for Clique/Indep. Set for any computable f

[Chen, Huang, Kanj, Xia – JCSS 2006]

- reduction from 3-coloring
  - split vertices into k groups of n/k vertices
  - create graph with k groups of $\le 3^{n/k}$ vertices, one per valid 3-coloring of input group
    - connect 2 colorings if they are compatible



check

  $\Rightarrow$ k-clique corresponds to 3-coloring
- if k-Clique solvable in $f(k)\, n^{k/s(k)}$ ← monotone increasing & unbounded

  then set k as large as possible such that $f(k) \le n$ & $k^{k/s(k)} \le n$

  $\Rightarrow$ $k = k(n)$ is unbounded function (min of 2 inverses)

  $\Rightarrow$ running time on reduced graph

$$
\begin{aligned}
&= f(k) \cdot (k\, 3^{n/k})^{k/s(k)} \\
&\le n \cdot k^{k/s(k)}\, 3^{n/s(k)} \\
&\le n^2 \cdot 3^{n/s(k(n))} \\
&= 2^{o(n)}
\end{aligned}
$$

solution to 3-coloring contradicting ETH

Parameterized reduction: $x \xrightarrow{f} x'$ (recall L13)

- parameter preserving: $k'(x') \le g(k(x))$ ← parameter blowup
- no $f(k)\, n^{o(k)}$ for A $\Rightarrow$ no $f'(k')\, n^{o(g^{-1}(k'))}$ for B
- e.g. no $f(k)\, n^{o(k)}$-time alg. for
  - Multicolored Clique/Indep. Set
  - Dominating Set, Set Cover

    } $k' = k$
  - Partial Vertex Cover (via better reduction)

Tool for parameterized complexity of planar problems:

Grid Tiling [Marx – FOCS 2007; ICALP 2012]

- given $k \times k$ grid, each cell (i, j) with set S_{ij} of 2D coordinates $\in \{1, 2, \dots, n\}$
- goal: choose one $x_{ij} \in S_{ij}$ $\forall i, j$ such that
  - vertical neighbors agree in first coordinate
  - horizontal neighbors agree in second coordinate

- W[1]-hard & ETH $\Rightarrow$ no $f(k)\, n^{o(k)}$-time algorithm
  - reduction from Clique, $V = \{v_1, v_2, \dots, v_n\}$
  - $k' = k$, $n' = n$
  - $S_{ii} = \{(v, v) \mid v \in V\}$ $\forall i$
  - $S_{ij} = \{(v, w) \in E \mid v \ne w\}$ $\forall i \ne j$

List coloring: given graph & list L_v of valid colors for each vertex v, is there a coloring?

- NP-hard even for planar & $|L_v| \le 3$ (3-coloring)
- parameterized by outerplanarity
  \# times can remove all vertices from outside face
  - $\in$ XP (bounded treewidth algorithm)
  - W[1]-hard & ETH $\Rightarrow$ no $f(k)\,n^{o(k)}$ algorithm
    - reduction from Grid Tiling
    - colors $= \{1, 2, \dots, n\}^2 \Rightarrow S_{ij}$ set of colors
    - $k \times k$ grid of vertices $u_{i,j}$, list $= S_{ij}$
    - between vertically adjacent vertices: vertex v_{ijcd}, list $= \{c, d\}$, connected to both $\forall$ colors c, d not agreeing on first coord.
      $\Rightarrow$ vertical neighbors agree on first coord.
      (if one uses c, v_{ijcd} used $d \Rightarrow d$ unavailable)
    - between horizontally adjacent vertices: vertex h_{ijcd}, list $= \{c, d\}$, connected to both $\forall$ colors c, d not agreeing on second coord.
      $\Rightarrow$ horizontal neighbors agree on second coord.
- by contrast: coloring is FPT w.r.t. outerplanarity (treewidth)

Grid tiling with $\le$:

- first coord$(x_{ij}) \le$ first coord$(x_{i+1,j})$ (column)
- second coord$(x_{ij}) \le$ second coord$(x_{i,j+1})$ (row)

- W[1]-hard & ETH $\Rightarrow$ no $f(k)n^{o(k)}$ algorithm
  - reduction from grid tiling
  - $k' = 4k$
  - 4×4 gadgets

Scattered set: (d-independent set)

- given graph & numbers k & d
- find k vertices with pairwise distances $\ge d$

- $d=2 \Rightarrow$ Independent Set $\Rightarrow$ W[1]-hard w.r.t. k
- planar graphs: FPT w.r.t. (k,d)
- planar graphs & d input:
  - $n^{O(\sqrt{k})}$-time algorithm
  - W[1]-hard w.r.t. k & ETH $\Rightarrow$ no $f(k)n^{o(\sqrt{k})}$ alg.
    - reduction from Grid Tiling with $\le$
    - $n \times n$ grid for grid cell (i,j)
    - color in $S_{ij} \to$ length $100D$ path attached to corresponding grid node
    - $d = 301n + 1$
    - $k' = k^2$

Unit-disk graphs:

- each vertex has coordinates in 2D ($\mathbb{Q}^2$)
- edge $\Leftrightarrow$ distance ≤ 1

- independent set = radius-$1/2$ disk packing with given centers
  - $n^{O(\sqrt{k})}$-time algorithm [Alber & Fiala – J. Alg. 2004]
  - W[1]-hard & no $f(k)\, n^{o(\sqrt{k})}$-time algorithm
    - reduction from Grid Tiling with $\leq$
    - $k \times k$ unit grid of $n \times n$ tiny grids of dots with only S_{ij} dots present
    - $k' = k^2$ (one per subgrid)
  - $\Rightarrow$ no EPTAS unless FPT = W[1]
  - ETH $\Rightarrow$ no $2^{(1/\varepsilon)^{O(1)}} n^{O((1/\varepsilon)^{1-\delta})}$ $(1+\varepsilon)$-approx. $\forall \delta$
    $\Rightarrow n^{O(1/\varepsilon)}$-time PTAS tight [Marx – FOCS 2007]