Lecture 3: Spherical Waves: Near & Far Field, Radiation Impedance, and Simple Sources

Suggested Reading: Fletcher pg., 100-109, Chapter 7; Kinsler et al. Chapter 8

I. Review of Wave Equations for Plane Waves

The one-dimensional wave equation for sound in a uniform plane wave,

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{\rho_0}{B} \frac{\partial^2 p(x,t)}{\partial t^2} \quad , \tag{3.1}$$

can be solved in terms of two plane waves traveling in opposite direction:

$$p(x,t) = f_+\left(t - x/c\right) + f_-\left(t + x/c\right) \quad , \tag{3.2}$$

$$v_x(x,t) = \frac{1}{z_0}\left[f_+\left(t - x/c\right) - f_-\left(t + x/c\right)\right] \text{ , and}$$

where: $z_0 = \sqrt{B_A \rho_0} = \rho_0 c$, and $c = \sqrt{B_A / \rho_0}$

Separating the time and space dependence in the sinusoidal steady state, where $k=\omega/c$:

$$p(x,t) = \text{Re}\left\{\underline{P}(x)e^{j\omega t}\right\} \quad v_x(x,t) = \text{Re}\left\{\underline{V}(x)e^{j\omega t}\right\}; \tag{3.3}$$

$$\underline{P}(x) = \underline{P}^+ e^{-jkx} + \underline{P}^- e^{jkx}; \quad \underline{V}(x) = \frac{1}{z_0}\left(\underline{P}^+ e^{-jkx} - \underline{P}^- e^{jkx}\right)$$

$$\bar{I} = \overline{p(t,x) v_x(t,x)} = \frac{1}{2}\text{Re}\left\{\underline{P}(x)\underline{V}^*(x)\right\} = \frac{1}{2}\frac{\left(\left|\underline{P}^+\right|^2 - \left|\underline{P}^-\right|^2\right)}{z_0} \quad , \tag{3.4}$$

where $\underline{P}^+$ and $\underline{P}^-$ are defined by the boundary conditions at the two ends of the one dimensional system.

In the case of a plane-wave propagating in an unbounded open space, there is only a wave traveling in one direction and therefore:

$$\underline{P}(x) = \underline{P}^+ e^{-jkx}; \quad \underline{V}(x) = \frac{\underline{P}^+ e^{-jkx}}{z_0}; \quad \bar{I} = \frac{1}{2}\frac{\left|\underline{P}^+\right|^2}{z_0} \text{ , and} \tag{3.5}$$

$$p(x,t) = \text{Re}\left\{\underline{P}(x)e^{j\omega t}\right\} = \left|\underline{P}^+\right| \cos\left(\omega t - kx + \angle \underline{P}^+\right) \tag{3.6}$$

Note that *P(x)* and *V(x)* are proportionally related by z_0 and that z_0 is real and independent of frequency.

We also saw that in one-dimensional systems with forward and backward waves the specific acoustic impedance varied in space and was complex:

$$\underline{Z}^S(x) = \frac{\underline{P}(x)}{\underline{V}_x(x)} \tag{3.7}$$

II. Spherically symmetric waves: Another kind of one-dimensional wave

A. Spherical Coordinates & Symmetry

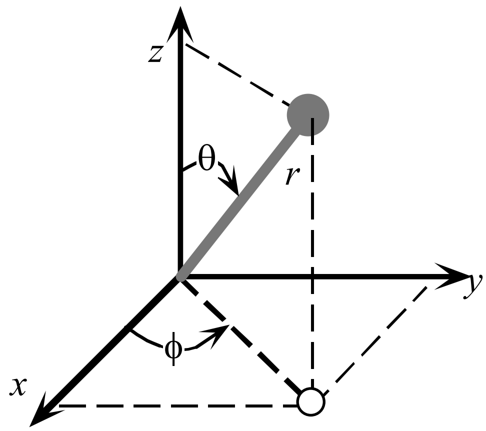


Fig. 3.1: The transformation between three dimensional Cartesian coordinates (*x,y,z*) and spherical coordinates *r*, θ, and φ.

If we assume "spherical symmetry" (i.e. the pressure and particle velocity only vary in *r*, the distance from the 'origin' of the spherical wave), then we can define a spherically symmetric wave equation:

$$\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial p(r,t)}{\partial r}\right)}{\partial r}=\frac{1}{c^2}\frac{\partial^2 p(r,t)}{\partial t^2} \quad . \tag{3.8}$$

We can make (3.8) analogous to the plane wave equation (3.1) by expressing both *p(r,t)* terms above as *r p(r,t)*: The first step is to multiply both sides of (3.8) by *r:*

$$r\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial p(r,t)}{\partial r}\right)}{\partial r}=r\frac{1}{c^2}\frac{\partial^2 p(r,t)}{\partial t^2} \quad ,$$

then rearrange such that

$$\frac{\partial^2 rp(r,t)}{\partial r^2}=\frac{1}{c^2}\frac{\partial^2 rp(r,t)}{\partial t^2} \quad . \tag{3.9}$$

This result is identical to (3.1) except that we have replaced *p(x,t)*, with *rp(r,t).*

Therefore a general solution to spherically symmetric waves is

$$rp(r,t)=f^+(t-r/c)+f^-(t+r/c), \text{ or}$$
$$p(r,t)=\frac{f^+(t-r/c)}{r}+\frac{f^-(t+r/c)}{r} \quad . \tag{3.10}$$

Note that this solution specifies that the amplitude of the pressure varies inversely with *r*. As *r* increases, the amplitude of the pressure falls. This is one difference between plane waves and spherical waves.

B. Spherical Waves in the Sinusoidal Steady State: Outward Wave Only

In the sinusoidal steady state:

$$p(r,t) = \mathrm{Re}\left\{\underline{P}(r)e^{j\omega t}\right\}, \tag{3.11}$$

$$\text{where:} \quad \underline{P}(r) = \frac{\underline{A}}{r}e^{-jkr}.$$

Comparing this description of the pressure term to the description of the outward going wave in a plane wave, $\underline{P}(x) = \underline{P}^{+}e^{-jkx}$, note that while the complex amplitude $\underline{P}^{+}$ has units of pressure, $\underline{A}$ has units of pressure times length.

What about velocity? We can relate the complex description of pressure and velocity using the acoustic version of Newtons' second law:

$$-\rho_0 \frac{\partial v_r(r,t)}{\partial t} = \frac{\partial p(r,t)}{\partial r} \quad . \tag{3.12}$$

leading to a more complicated solution for the velocity where

$$v(r,t) = \mathrm{Re}\left\{\underline{V}(r)e^{j\omega t}\right\}, \text{ and} \tag{3.13}$$

$$\underline{V}(r) = \underline{P}(r)\frac{1}{\rho_0 c}\left(1 + \frac{1}{jkr}\right) = \frac{\underline{A}}{r}e^{-jkr}\frac{1}{\rho_0 c}\left(1 + \frac{1}{jkr}\right) \quad . \tag{3.14}$$

The particle velocity, for a given source amplitude, <u>varies with distance from the source *r*, in a nonlinear manner and also varies wave number *k* *(unlike uniform plane waves)*</u>.

The specific acoustic impedance (magnitude and angle) seen by the wave depends on *r* and *k*:

$$\underline{Z}^{S}(r) = \frac{\underline{P}(r)}{\underline{V}(r)} = \frac{\rho_0 c}{1 + \frac{1}{jkr}} = \frac{z_0}{1 + \frac{1}{jkr}} \quad . \tag{3.15}$$

Equations 3.11, 3.14 and 3.15 define *$\underline{P}(r)$, $\underline{V}(r)$* and *$\underline{Z}^{S}(r)$* at all distances from the wave source, but, some useful approximations work <u>near to</u> and <u>far from</u> the source.

In the "<u>Far Field</u>", where $kr >> 1$, Equations 3.16 and 3.17 are greatly simplified:

$$\underline{Z}^{S}(r)\Big|_{kr>>1} \approx \rho_0 c \quad \text{and} \quad \underline{V}(r)\Big|_{kr>>1} \approx \frac{\underline{A}}{r}e^{-jkr}\frac{1}{\rho_0 c} = \frac{\underline{P}(r)}{\rho_0 c} \quad . \tag{3.18a\&b}$$

> In the "<u>Far Field</u>" *$\underline{V}(r)$* and *$\underline{P}(r)$* are proportionately related by the characteristic impedance of the medium *$z_0 = \rho_0 c$ as in a uniform plane wave*, and the magnitudes of *$\underline{V}(r)$* and *$\underline{P}(r)$* decrease proportionately with distance from the source.

The inverse proportionality between *r* and $|\underline{V}(r)|$ and $|\underline{P}(r)|$ leads to an average power density (or sound intensity) that decreases as the square of *r*:

$$\bar{I}(r)\Big|_{kr>>1} = \frac{1}{2}\mathrm{Re}\left\{\underline{P}(r)\underline{V}^{*}(r)\right\} \approx \frac{1}{2}\frac{|\underline{P}(r)|^2}{z_0} = \frac{1}{2}|\underline{V}(r)|^2 z_0 = \frac{1}{2}\frac{|\underline{A}|^2}{z_0 r^2}. \tag{3.19}$$

This relationship is often referred to as the "inverse square law".

In the "Near Field" where $kr<<1$, $\underline{Z}^S(r)$ is approximately masslike:

$$\underline{Z}^S(r) = \frac{z_0}{1+\frac{1}{jkr}} = \frac{jz_0 kr}{jkr+1}$$

$$\underline{Z}^S(r)\Big|_{kr<<1} \approx jz_0 kr \ , \qquad (3.20)$$

and the particle velocity lags the sound pressure by π/2 radians:

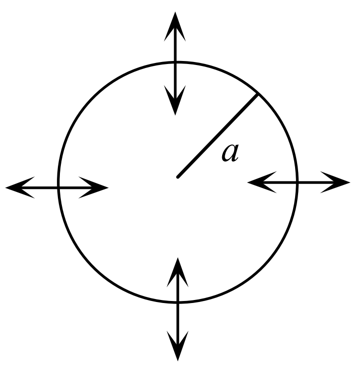
$$\underline{V}(r)\Big|_{kr<<1} = \frac{\underline{P}(r)}{\underline{Z}^S} \approx \frac{\underline{P}(r)}{jz_0 kr} . \qquad (3.21)$$

Since $\underline{Z}^S(r)$ is dominated by a reactive term when $kr<<1$, little power is transferred from the source to the space that surrounds it.

2. "Simple" Spherical Sources:

A. Pulsing Sphere

Fig 3.2 A pulsing sphere



Where simple means all parts of the surface are vibrating in phase! The sphere pulsations are also constrained to be small compared to the steady-state dimensions.

"Source Strength"= $\underline{U}_S = 4\pi a^2 \underline{V}(a)$. (3.22)

(Note that Source Strength is a volume velocity.)

The "Radiation impedance" (with units of Acoustic Ohms Pa-s/ m^3) at the surface of the source is:

$$\underline{Z}(a) = \frac{\underline{P}(a)}{\underline{U}_S} = \frac{1}{4\pi a^2}\frac{z_0}{1+\frac{1}{jka}} = \frac{z_0}{4\pi a^2}\frac{jka}{1+jka} . \qquad (3.23)$$

In the High Frequencies, $ka >>1$, $\underline{Z}(a)$ looks like a characteristic *acoustic* impedance:

$$\underline{Z}(a) \approx \frac{z_0}{4\pi a^2} . \qquad (3.24)$$

At Low Frequencies, when $ka <<1$, $\frac{1}{1+jka} \approx (1-jka)$, and

$$\underline{Z}(a) \approx \frac{z_0}{4\pi a^2}(1-jka)jka = \frac{z_0}{4\pi a^2}\left((ka)^2 + jka\right). \qquad (3.25)$$

Since the real part of $\underline{Z}(a)$ in the "Low Frequencies" is proportional to ω^2, the average power radiated for a given source strength is also proportional to frequency.

$$\overline{\Pi} = \frac{1}{2}\left|\underline{U}_S\right|^2 \mathrm{Re}\{\underline{Z}(a)\} \approx \left|\underline{U}_S\right|^2 \frac{z_0 (ka)^2}{8\pi a^2} = \left|\underline{U}_S\right|^2 \frac{\omega^2 z_0}{8\pi c^2} \quad . \tag{3.26}$$

i.e. at low *ka*, <u>little average sound power radiates to the environment</u>. Also note <u>that for a given $\underline{U}_S$,</u> with *ka* <<1 the average power radiated is independent of the dimensions of the source!

How can we describe wave propagation from a spherical source in terms of source strength? We have described the sound pressure in a spherical wave in terms of a complex constant $\underline{A}$, i.e.

$$\underline{P}(r) = \frac{\underline{A}}{r} e^{-jkr} \quad .$$

Knowing the source strength, we can define $\underline{A}$ in terms of the sound pressure at the walls of the spherical source:

$$\underline{P}(a) = \frac{\underline{A}}{a} e^{-jka} = \underline{U}_S \underline{Z}(a); \text{ i.e. } \underline{A} = a\underline{U}_S \underline{Z}(a) e^{-jka} \quad . \tag{3.27}$$

In the Low Frequency situation, i.e. *ka* << 1, we can approximate e^{-jka} as 1, and we can use the Low Frequency approximation for $\underline{Z}(a)$:

$$\underline{A} \approx a\underline{U}_S \frac{z_0}{4\pi a^2}\left((ka)^2 + jka\right) \approx j\omega \underline{U}_S \frac{\rho_0}{4\pi} \quad . \tag{3.28}$$

Therefore, when the <u>radius</u> *a* of the source is such that that *ka* << 1, the sound pressure at some <u>distance</u> *r* is:

$$\underline{P}(r) = j\omega \underline{U}_S \frac{\rho_0}{4\pi r} e^{-jkr} \quad . \tag{3.29}$$

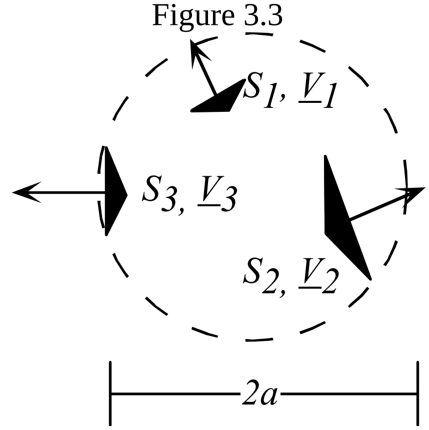
3. Generalization of the simple source concept

The sound radiated from an acoustically small source with *kx* << 1, where *x* is some descriptive linear dimension of the source, can be characterized by a source strength U_S as long as all parts of the 'radiator' move in phase.

For example the output of three small loud speakers - of diaphragm areas S_1, S_2 and S_3 and diaphragm velocities $\underline{V}_1$, $\underline{V}_2$, $\underline{V}_3$, that all fit within an imaginary sphere of radius *a* can be approximated by the output of a simple source with source strength:

$$\underline{U}_S = \sum_{i=1}^{n} S_i \underline{V}_i \quad . \tag{3.30}$$
$$= \iint_S \overline{V}(S) \bullet dS$$

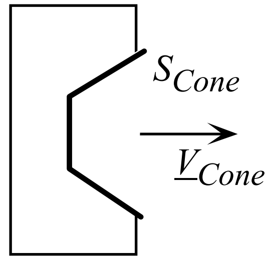
Figure 3.3



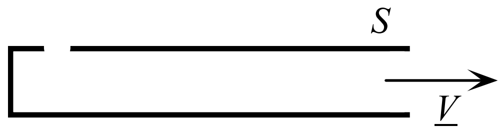
<u>As long as all parts of all of the diaphragms are moving in phase and *ka* <<1.</u>

Other low-frequency "simple sources" include:
Figure 3.4

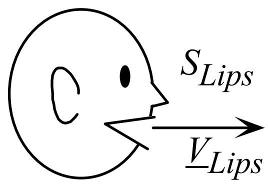
A Loud Speaker in a box,



The open end of an organ pipe,



Radiation from the mouth.



The equivalence to a simple source when ka <<1 also implies that far away from the radiator in the *Far Field*, where $r >> a$, the radiation is spherically symmetric and the sound pressures and particle velocities within the wave are quantifiable in terms of the source strength and the *Far Field* produced by a spherical source:

$$\underline{P}(r) \approx \frac{\underline{A}}{r} e^{-jkr}, \quad \underline{V}(r) \approx \frac{\underline{A}}{\rho_0 c r} e^{-jkr}, \text{ where } \underline{A}\big|_{ka<<1} \approx j\omega \underline{U}_S \frac{\rho_0}{4\pi} \text{ and } \underline{z}(r) \approx z_0 \ . \qquad (3.31)$$

4. More About Radiation Impedance

We have just argued that the specific acoustic impedance which describes the relationship between sound pressure and particle velocity is the same in the far field for any 'simple' source. However, one constraint on sound radiation that differs for the four simple sources in Figures 3.3 and 3.4 is the load that the surrounding air places on the radiators, i.e. the radiation impedance $\underline{Z}_R$. Knowledge of $\underline{Z}_R$ allows us to quantify:

(1). Power radiated from a source to the environment, and
(2). The resistive and reactive forces of the medium on the source.

The pulsing sphere revisited:

We have already derived the radiation impedance acting on the surface of a pulsing sphere of radius a, where we can modify (3.23) such that:

$$\underline{Z}_R = \frac{\underline{P}(a)}{\underline{U}_S} = \frac{z_0}{4\pi a^2} \frac{jka}{1+jka}\left(\frac{1-jka}{1-jka}\right) = \frac{z_0}{4\pi a^2} \frac{(ka)^2 + jka}{(ka)^2+1} \qquad (3.32)$$

Eqn. (3.32 describes a real part and an imaginary part to $\underline{Z}_R$. where

$$R_R = \frac{z_0 (ka)^2}{4\pi a^2\left((ka)^2+1\right)}, \text{ and } X_R = \frac{z_0 ka}{4\pi a^2\left((ka)^2+1\right)} \ . \qquad (3.33)$$

According to (3.33), at low frequencies when $ka << 1$, the radiation resistance is independent of the sphere's radius and has a magnitude that increases as the square of ω:

$$R_R\big|_{ka<<1} \approx \frac{z_0(ka)^2}{4\pi a^2} = \frac{z_0 k^2}{4\pi} = \frac{z_0\omega^2}{4\pi c^2} \quad . \tag{3.34}$$

In the same low-frequency range, the radiation reactance is positive and proportional to frequency and is well approximated by an acoustic mass or inertance:

$$X_R\big|_{ka<<1} \approx \frac{z_0 ka}{4\pi a^2} = \omega\frac{\rho_0 a}{4\pi a^2} = \omega M \quad . \tag{3.35}$$

This mass is equivalent to a blanket of air around the sphere of thickness a.

At high frequencies, $ka >> 1$,the radiation resistance approximates the ratio of the characteristic impedance of the medium and the area of the sphere and the reactance decreases proportionately with sound frequency:

$$R_R\big|_{ka>>1} \approx \frac{z_0}{4\pi a^2}, \text{ and } X_R\big|_{ka>>1} \approx \frac{z_0}{4\pi a^2 ka} = \frac{\rho_0 c^2}{4\pi a^3\omega} \quad . \tag{3.36}$$

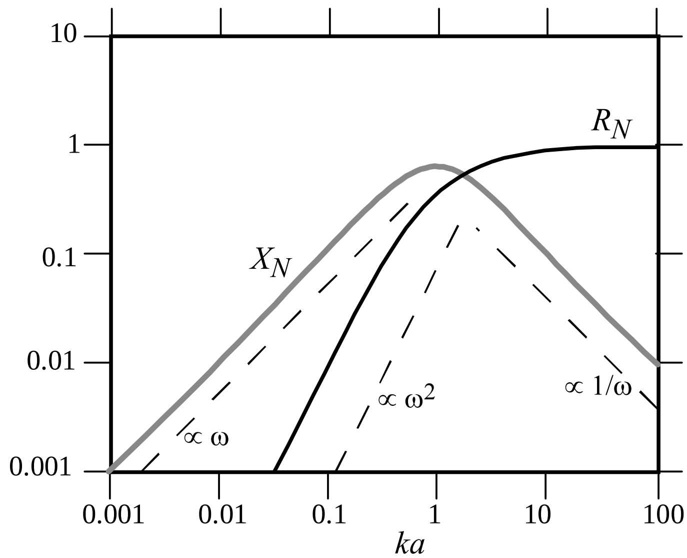


Figure 3.5: The normalized radiation resistance R_N and reactance X_N acting on a pulsating sphere. The normalization factor depends on the surface area of the sphere S and the characteristic impedance of the media z_0. The dashed lines illustrate the slopes of relationships that are proportional to ω, ω^2 and 1/ω.

Each of the impedance components described above have a non-simple frequency dependence. There is trick to thinking about these in a more simple way. The *radiation admittance* of a sphere is much simpler in form, where

$$\underline{Y}_R = \frac{1}{\underline{Z}_R} = \frac{1}{R_{YR}} + \frac{1}{X_{YR}},$$

$$\text{where: } R_{YR} = \frac{z_0}{4\pi a^2}, \text{ and } X_{YR} \approx \omega\frac{\rho_0 a}{4\pi a^2}$$

More discussions of the radiation impedance can be found in Beranek 1986.

4. Combinations of Simple Sources

Source - frequency combinations that do not meet either the small *ka* or *"in-phase"* requirements can sometimes be approximated by combinations of simple sources. For example, if we are concerned about the far-field transmission from the lips of sound frequencies whose wave lengths approximate the mouth opening $ka \approx 1$, you could model the mouth as an array of simple sources

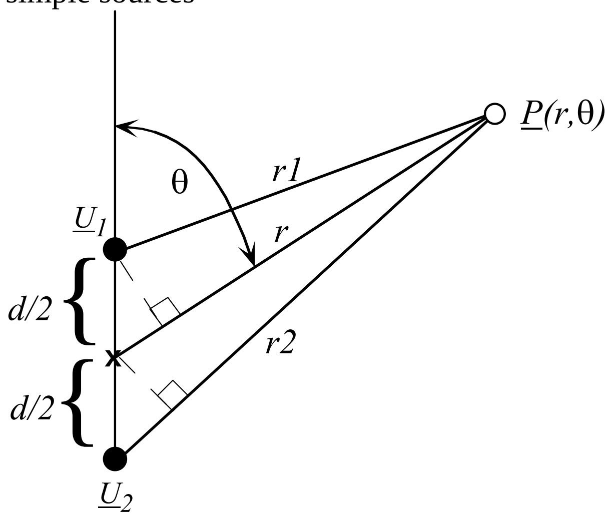


Figure 3.6 Two simple sources $\underline{U}_1$ and $\underline{U}_2$ are separated by a distance *d*. We are interested in the sound pressure $\underline{P}(r,\theta)$ at a point in the far field ($r >> d$, the open circle). The distance between the measurement point and the two sources is *r1* and *r2*. *r is the* distance between the measurement point and a point half-way between the two sources (the x). θ is the angle between *r* and the line defined by the two sources.

Using superposition:

$$\underline{P}(r,\theta) = \underline{P}_1(r) + \underline{P}_2(r) = \frac{j\omega\rho_0}{4\pi}\left(\frac{\underline{U}_1}{r_1}e^{-jkr_1} + \frac{\underline{U}_2}{r_2}e^{-jkr_2}\right). \tag{3.37}$$

Since $r >> d$ we can assume r, $r1$ and $r2$ are parallel such that $r1 \approx r - \frac{d}{2}\cos\theta$ and $r2 \approx r + \frac{d}{2}\cos\theta$:

$$\underline{P}(r,\theta) = \frac{j\omega\rho_0}{4\pi}\left(\frac{\underline{U}_1}{r-(d/2)\cos\theta}e^{-jk(r-(d/2)\cos\theta)} + \frac{\underline{U}_2}{r+(d/2)\cos\theta}e^{-jk(r+(d/2)\cos\theta)}\right)$$

Furthermore, since $r >> (d/2)\cos\theta$, the effect of distance on the magnitudes of each term are approximately equal and can be factored out along with the common e^{-jkr} dependence:

$$\underline{P}(r,\theta) = \frac{j\omega\rho_0}{4\pi r}e^{-jkr}\left(\underline{U}_1 e^{+jk(d/2)\cos\theta} + \underline{U}_2 e^{-jk(d/2)\cos\theta}\right). \tag{3.38}$$

Finally, for the special case where $|\underline{U}_1| = |\underline{U}_2| = U_0$ and $\angle\underline{U}_2 - \angle\underline{U}_1 = \phi$:

$$\underline{P}(r,\theta) = \frac{j\omega\rho_0}{2\pi r}U_0 e^{-jkr}\cos\left(k(d/2)\cos\theta + \phi/2\right). \tag{3.39}$$

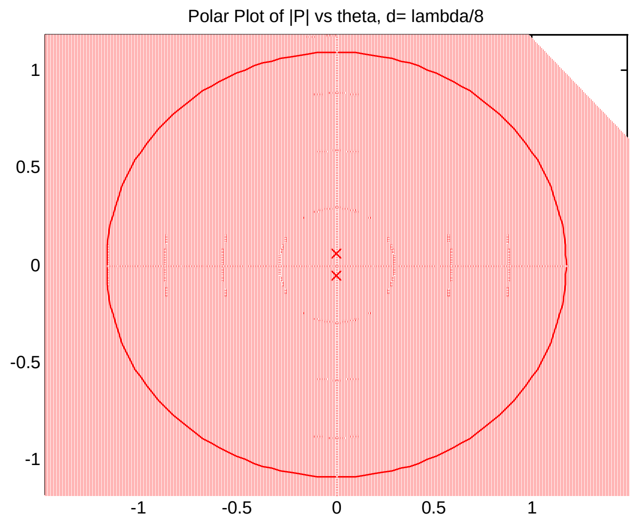
-The multiplier to the cosine function in 3.39 defines an equivalent simple source of strength $\frac{j\omega\rho_0}{2\pi r}U_0$ and propagation constant e^{-jkr}.

-The cosine function $\cos\left(k(d/2)\cos\theta + \phi/2\right)$ defines a directionality to the source output that depends on k, d, θ and ϕ.

5. Output of Arrays

Case A. Two simple sources in phase and of equal source strength

Equation 3.39 is relevant, $\underline{P}(r,\theta) = \frac{j\omega\rho_0}{2\pi r} U_0 e^{-jkr} g(\theta)$, where $g(\theta) = \cos\left(k(d/2)\cos\theta + \phi/2\right)$ and $\phi = 0$;



Example 1: Sources are separated by a distance $d = \lambda/8$. What's kd?

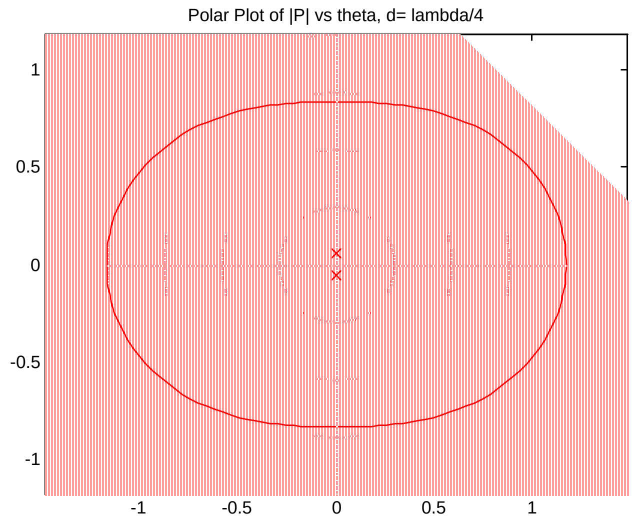
The plot on the left is a polar plot of the variation in $|\underline{P}|$ vs. θ at a large distance from the source. The 'x's show the source axis. The vertical dotted line shows the direction 'in-line' with the sources. The horizontal line is the direction perpendicular to the source line. The concentric circles code pressure amplitude as a function of θ. With $d=\lambda/8$, The sound pressure magnitude is nearly nondirectional.

How can we think about the small reductions in |P| that do occur about the angles that are close to on-axis? Why are the pressures that are off-axis larger in magnitude?

$g(\theta=0)=\cos(2\pi/\lambda\ d/2\ \cos(0))=\cos(2\pi/\lambda\ \lambda/16\ \cos(0))=\cos(\pi/8)$

$g(\theta=\pi/2)=\cos(\pi/8\ \cos(\pi/2))=$_______

Example 2: $d=\lambda/4$



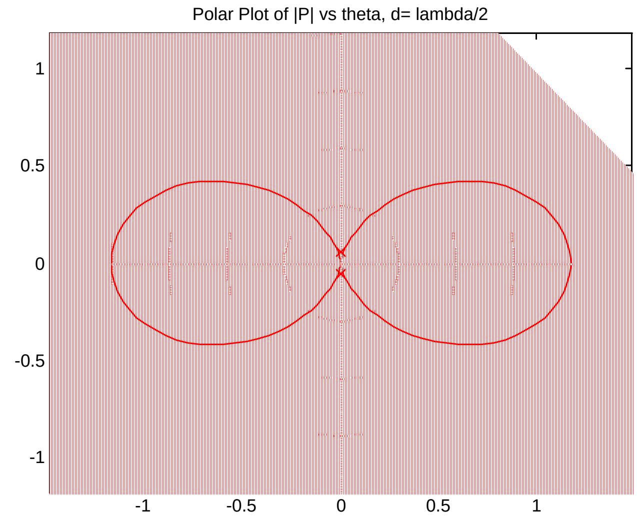
$g(0)=\cos(2\pi/\lambda\ \lambda/8\ \cos(0))=$ ______

$g(\pi/2)=\cos(\pi/4\ \cos(\pi/2)) =$ ______

<u>Example 3</u>: $d=\lambda/2$

$g(0)=\cos(2\pi/\lambda\ \lambda/4\ \cos(0))=$ ____

$g(\pi/2)=\cos(\pi/2\ \cos(\pi/2))=$ ______

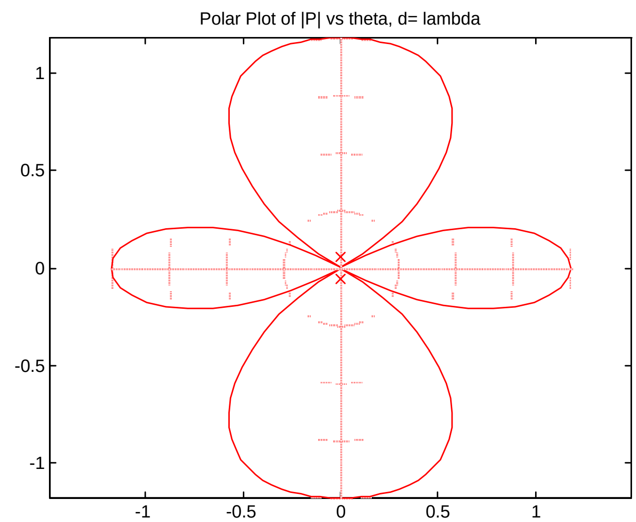


<u>Example 4</u>: $d=\lambda$

$g(0)=\cos(2\pi/\lambda\ \lambda/2\ \cos(0))=$ _____

$g(\pi/2)=\cos(\pi\ \cos(\pi/2))=$ _______

$g(\pi/3)=\cos(\pi\ \cos(\pi/3))=$ ________

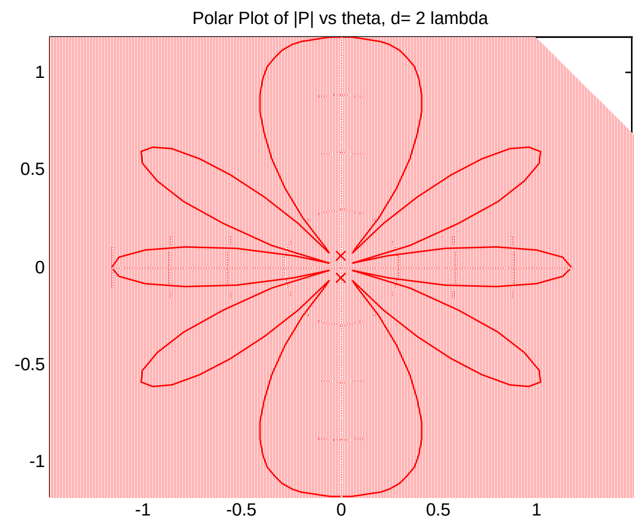


<u>Example 5:</u> $d=2\lambda$

g(?) = 0?

What do these patterns look like in three-dimensions?

Are there some simple rules to the number of nodes in each pattern?



B. Equal strength Sources that are close to each other and out of phase: The dipole

$\underline{U}_1 = -\underline{U}_2$; therefore $|\underline{U}_1|=|\underline{U}_2|=U_0$ but $\angle\underline{U}_1-\angle\underline{U}_2=\pi$; and $\phi/2=\pi/2$;

$$\underline{P}(r,\theta)=\frac{j\omega\rho_0}{2\pi r}U_0 e^{-jkr}\cos\left(k\,d/2\cos\theta+\phi/2\right) \tag{3.40}$$

Since $\phi/2=\pi/2$, and $k=2\pi/\lambda$;

$$\underline{P}(r,\theta)=\frac{j\omega\rho_0}{2\pi r}U_0 e^{-jkr}\sin\left(d\pi/\lambda\cos\theta\right)\quad . \tag{3.41}$$

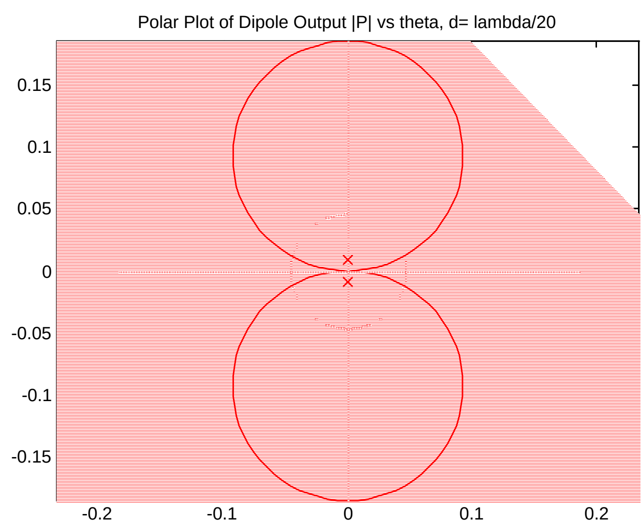
Finally since $d \ll \lambda$;

$$\underline{P}(r,\theta)=\frac{j\omega\rho_0}{2\pi r}U_0 e^{-jkr}\;\pi/\lambda d\cos\theta=\frac{j\omega^2\rho_0}{4\pi rc}dU_0\cos\theta e^{-jkr}\quad . \tag{3.42}$$

where $|\underline{P}(r,\theta)|$ depends directly on d, U_0, ω^2, $1/r$ and $\cos\theta$. The product dU_0 is sometimes called *"Dipole Strength"*.

The dipole has a directivity pattern (on the right) that in one dimension is similar to the reverse of the two in-phase simple sources with $d=\lambda/2$. Are they similar in three dimensions? __________

Also notice the difference in the amplitude of the pressures between here and Example 3 above.



In example 3 the maximum pressure magnitude is measured with $\theta=\pm\pi/2$: $\left|\underline{P}_{MAX}\right|^{Monopole}=\frac{\rho_0\omega U_0}{2\pi r}$. The maximum pressure amplitude produced by the dipole is:

$$\left|\underline{P}_{MAX}\right|^{Dipole}=\frac{dU_0\omega^2\rho_0}{4\pi rc}\text{ , such that} \tag{3.43}$$

$$\frac{\left|\underline{P}_{MAX}\right|^{Dipole}}{\left|\underline{P}_{MAX}\right|^{Monopole}}=\frac{\dfrac{dU_0\omega^2\rho_0}{4\pi rc}}{\dfrac{\rho_0\omega U_0}{2\pi r}}=\frac{d\omega}{2c}=\pi\frac{d}{\lambda}<1\quad . \tag{3.44}$$

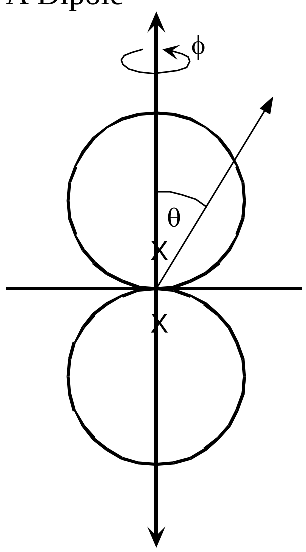
C. General Conclusions:

Arrays of simple sources can produce radiation patterns that are directional.

The directivity depends on the spacing and the phases of the sources.

D. Directivity Index

A Dipole



A useful metric that characterizes and quantifies angular selectivity (or directivity) of a sources output, is the *Directivity Index., D,* where:

$$D = \frac{\text{Mean - Square response in some reference direction}}{\text{Mean Square response averaged over all angles}},$$

or

$$D = \frac{|\mathrm{Pr}ef|^2}{\frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}|P(r,\theta,\phi)|^2 \sin\theta d\theta d\phi}$$

where the reference direction is usually the axis describing the largest response. For example the axis of an acoustic dipole is defined by the line that connects the two sources and the two maxima in pressure response. A suitable reference angle is θ=0. In the dipole $|\underline{P}(r,\theta,\phi)| \propto \cos\theta$, and

$$D = \frac{\cos^2(0)}{\frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\cos^2\theta\sin\theta d\theta d\phi} = \frac{1}{\frac{1}{4\pi}\frac{4\pi}{3}} = 3,$$

where

$$10\log_{10}(3) = 4.8\,dB.$$

In an acoustic monopole or simple source where the output is non-directional, i.e. $|\underline{P}(r,\theta,\phi)|$ is independent of direction:

$$D = \frac{|P(r)|^2}{\frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}|P(r)|^2 \sin\theta d\theta d\phi} = \frac{1}{\frac{1}{4\pi}4\pi} = 1 \quad .$$