6.231 DYNAMIC PROGRAMMING

LECTURE 5

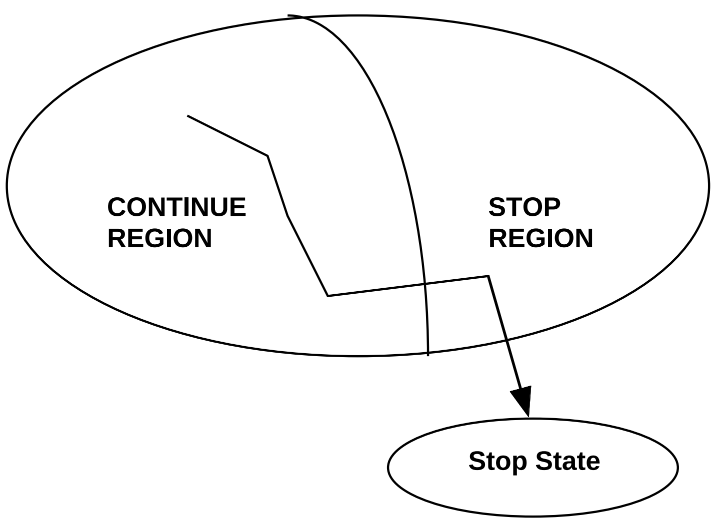
LECTURE OUTLINE

- Stopping problems
- Scheduling problems
- Minimax Control

PURE STOPPING PROBLEMS

- Two possible controls:
  - Stop (incur a one-time stopping cost, and move to cost-free and absorbing stop state)
  - Continue [using $x_{k+1} = f_k(x_k, w_k)$ and incurring the cost-per-stage]

- Each policy consists of a partition of the set of states x_k into two regions:
  - Stop region, where we stop
  - Continue region, where we continue



EXAMPLE: ASSET SELLING

- A person has an asset, and at $k = 0, 1, \ldots, N-1$ receives a random offer w_k
- May accept w_k and invest the money at fixed rate of interest r, or reject w_k and wait for w_{k+1}. Must accept the last offer w_{N-1}
- DP algorithm (x_k: current offer, T: stop state):

$$J_N(x_N) = \begin{cases} x_N & \text{if } x_N \neq T, \\ 0 & \text{if } x_N = T, \end{cases}$$

$$J_k(x_k) = \begin{cases} \max\Big[(1+r)^{N-k} x_k,\ E\big\{J_{k+1}(w_k)\big\}\Big] & \text{if } x_k \neq T, \\ 0 & \text{if } x_k = T. \end{cases}$$

- Optimal policy;

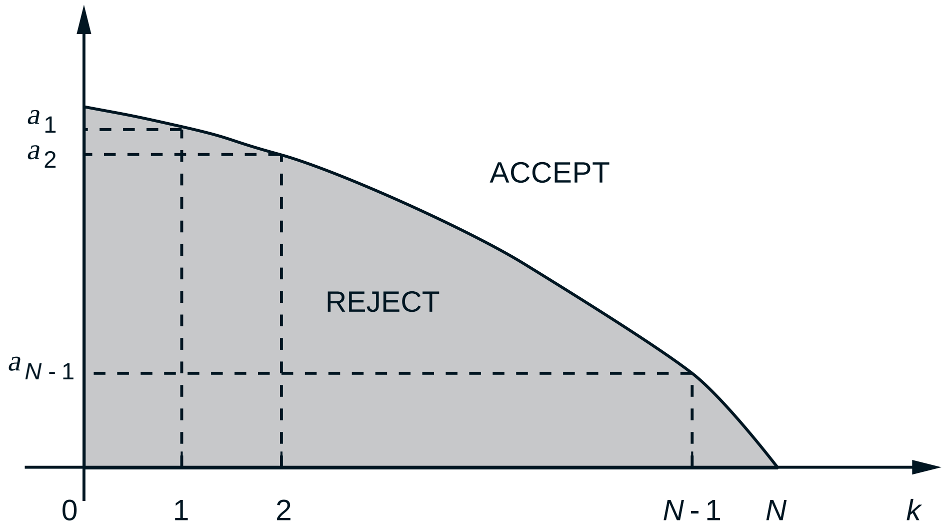
$$\text{accept the offer } x_k \qquad \text{if } x_k > \alpha_k,$$

$$\text{reject the offer } x_k \qquad \text{if } x_k < \alpha_k,$$

where

$$\alpha_k = \frac{E\big\{J_{k+1}(w_k)\big\}}{(1+r)^{N-k}}.$$

FURTHER ANALYSIS



- Can show that $\alpha_k \geq \alpha_{k+1}$ for all k

- Proof: Let $V_k(x_k) = J_k(x_k)/(1+r)^{N-k}$ for $x_k \neq T$. Then the DP algorithm is

$$V_N(x_N) = x_N, \;\; V_k(x_k) = \max\left[x_k,\, (1+r)^{-1} \mathop{E}_{w}\left\{V_{k+1}(w)\right\}\right]$$

We have $\alpha_k = E_w\left\{V_{k+1}(w)\right\}/(1+r)$, so it is enough to show that $V_k(x) \geq V_{k+1}(x)$ for all x and k. Start with $V_{N-1}(x) \geq V_N(x)$ and use the monotonicity property of DP. **Q.E.D.**

- We can also show that if w is bounded, $\alpha_k \to \overline{a}$ as $k \to -\infty$. Suggests that for an infinite horizon the optimal policy is stationary.

GENERAL STOPPING PROBLEMS

- At time k, we may stop at cost $t(x_k)$ or choose a control $u_k \in U(x_k)$ and continue

$$J_N(x_N) = t(x_N),$$

$$J_k(x_k) = \min\Big[t(x_k), \min_{u_k \in U(x_k)} E\Big\{g(x_k, u_k, w_k) + J_{k+1}\big(f(x_k, u_k, w_k)\big)\Big\}\Big]$$

- Optimal to stop at time k for x in the set

$$T_k = \left\{ x \;\Big|\; t(x) \le \min_{u \in U(x)} E\Big\{g(x, u, w) + J_{k+1}\big(f(x, u, w)\big)\Big\} \right\}$$

- Since $J_{N-1}(x) \le J_N(x)$, we have $J_k(x) \le J_{k+1}(x)$ for all k, so

$$T_0 \subset \cdots \subset T_k \subset T_{k+1} \subset \cdots \subset T_{N-1}.$$

- Interesting case is when all the T_k are equal (to T_{N-1}, the set where it is better to stop than to go one step and stop). Can be shown to be true if

$$f(x, u, w) \in T_{N-1}, \qquad \text{for all } x \in T_{N-1},\ u \in U(x),\ w.$$

SCHEDULING PROBLEMS

- We have a set of tasks to perform, the ordering is subject to optimal choice.
- Costs depend on the order
- There may be stochastic uncertainty, and precedence and resource availability constraints
- Some of the hardest combinatorial problems are of this type (e.g., traveling salesman, vehicle routing, etc.)
- Some special problems admit a simple quasi-analytical solution method
  - Optimal policy has an "index form", i.e., each task has an easily calculable "cost index", and it is optimal to select the task that has the minimum value of index (multi-armed bandit problems - to be discussed later)
  - Some problems can be solved by an "interchange argument" (start with some schedule, interchange two adjacent tasks, and see what happens). They require existence of an optimal policy which is open-loop.

EXAMPLE: THE QUIZ PROBLEM

- Given a list of N questions. If question i is answered correctly (given probability p_i), we receive reward R_i; if not the quiz terminates. Choose order of questions to maximize expected reward.
- Let i and j be the kth and $(k+1)$st questions in an optimally ordered list

$$L = (i_0, \dots, i_{k-1}, i, j, i_{k+2}, \dots, i_{N-1})$$

$$\begin{aligned} E\,\{\text{reward of } L\} = E\Big\{&\text{reward of } \ \{i_0, \dots, i_{k-1}\}\Big\} \\ &+ p_{i_0} \cdots p_{i_{k-1}} (p_i R_i + p_i p_j R_j) \\ &+ p_{i_0} \cdots p_{i_{k-1}} p_i p_j E\Big\{\text{reward of } \{i_{k+2}, \dots, i_{N-1}\}\Big\} \end{aligned}$$

Consider the list with i and j interchanged

$$L' = (i_0, \dots, i_{k-1}, j, i, i_{k+2}, \dots, i_{N-1})$$

Since L is optimal, $E\{\text{reward of } L\} \geq E\{\text{reward of } L'\}$, so it follows that $p_i R_i + p_i p_j R_j \geq p_j R_j + p_j p_i R_i$ or

$$p_i R_i / (1 - p_i) \geq p_j R_j / (1 - p_j).$$

MINIMAX CONTROL

- Consider basic problem with the difference that the disturbance w_k instead of being random, it is just known to belong to a given set $W_k(x_k, u_k)$.

- Find policy π that minimizes the cost

$$J_\pi(x_0) = \max_{\substack{w_k \in W_k(x_k, \mu_k(x_k)) \\ k=0,1,\ldots,N-1}} \left[g_N(x_N) + \sum_{k=0}^{N-1} g_k\big(x_k, \mu_k(x_k), w_k\big) \right]$$

- The DP algorithm takes the form

$$J_N(x_N) = g_N(x_N),$$

$$J_k(x_k) = \min_{u_k \in U(x_k)} \max_{w_k \in W_k(x_k, u_k)} \Big[g_k(x_k, u_k, w_k) + J_{k+1}\big(f_k(x_k, u_k, w_k)\big) \Big]$$

(Section 1.6 in the text).

DERIVATION OF MINIMAX DP ALGORITHM

- Similar to the DP algorithm for stochastic problems. The optimal cost $J^*(x_0)$ is

$$J^*(x_0) = \min_{\mu_0} \cdots \min_{\mu_{N-1}} \max_{w_0 \in W[x_0, \mu_0(x_0)]} \cdots \max_{w_{N-1} \in W[x_{N-1}, \mu_{N-1}(x_{N-1})]} \left[\sum_{k=0}^{N-1} g_k\big(x_k, \mu_k(x_k), w_k\big) + g_N(x_N) \right]$$

$$= \min_{\mu_0} \cdots \min_{\mu_{N-2}} \Bigg[\min_{\mu_{N-1}} \max_{w_0 \in W[x_0, \mu_0(x_0)]} \cdots \max_{w_{N-2} \in W[x_{N-2}, \mu_{N-2}(x_{N-2})]} \left[\sum_{k=0}^{N-2} g_k\big(x_k, \mu_k(x_k), w_k\big) + \max_{w_{N-1} \in W[x_{N-1}, \mu_{N-1}(x_{N-1})]} \Big[g_{N-1}\big(x_{N-1}, \mu_{N-1}(x_{N-1}), w_{N-1}\big) + J_N(x_N) \Big] \right] \Bigg]$$

- Interchange the min over μ_{N-1} and the max over $w_0, \ldots, w_{N-2}$, and similarly continue backwards, with $N - 1$ in place of N, etc. After N steps we obtain $J^*(x_0) = J_0(x_0)$.

- Construct optimal policy by minimizing in the RHS of the DP algorithm.

UNKNOWN-BUT-BOUNDED CONTROL

- For each k, keep the x_k of the controlled system

$$x_{k+1} = f_k\big(x_k, \mu_k(x_k), w_k\big)$$

inside a given set X_k, the *target set at time* k.

- This is a minimax control problem, where the cost at stage k is

$$g_k(x_k) = \begin{cases} 0 & \text{if } x_k \in X_k, \\ 1 & \text{if } x_k \notin X_k. \end{cases}$$

- We must reach at time k the set

$$\overline{X}_k = \big\{x_k \mid J_k(x_k) = 0\big\}$$

in order to be able to maintain the state within the subsequent target sets.

- Start with $\overline{X}_N = X_N$, and for $k = 0, 1, \ldots, N-1$,

$$\overline{X}_k = \Big\{x_k \in X_k \mid \text{ there exists } u_k \in U_k(x_k) \text{ such that } f_k(x_k, u_k, w_k) \in \overline{X}_{k+1}, \text{ for all } w_k \in W_k(x_k, u_k)\Big\}$$

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