Homework 6 additional problems

1. *Maximizing house profit in a gamble and imputed probabilities.* A set of n participants bet on which one of m outcomes, labeled $1, \ldots, m$, will occur. Participant i offers to purchase up to $q_i > 0$ gambling contracts, at price $p_i > 0$, that the true outcome will be in the set $S_i \subset \{1, \ldots, m\}$. The house then sells her x_i contracts, with $0 \leq x_i \leq q_i$. If the true outcome j is in S_i, then participant i receives \$1 per contract, *i.e.*, x_i. Otherwise, she loses, and receives nothing. The house collects a total of $x_1 p_1 + \cdots + x_n p_n$, and pays out an amount that depends on the outcome j,

$$\sum_{j \in S_i} x_i.$$

The difference is the house profit.

(a) *Optimal house strategy.* How should the house decide on x so that its worst-case profit (over the possible outcomes) is maximized? (The house determines x after examining all the participant offers.)

(b) *Imputed probabilities.* Suppose $x^\star$ maximizes the worst-case house profit. Show that there exists a probability distribution π on the possible outcomes (*i.e.*, $\pi \in \mathbf{R}_+^m$, $\mathbf{1}^T \pi = 1$) for which $x^\star$ also maximizes the expected house profit. Explain how to find π.

Hint. Formulate the problem in part (a) as an LP; you can construct π from optimal dual variables for this LP.

Remark. Given π, the 'fair' price for offer i is $p_i^{\text{fair}} = \sum_{j \in S_i} \pi_j$. All offers with $p_i > p_i^{\text{fair}}$ will be completely filled (*i.e.*, $x_i = q_i$); all offers with $p_i < p_i^{\text{fair}}$ will be rejected (*i.e.*, $x_i = 0$).

Remark. This exercise shows how the probabilities of outcomes (*e.g.*, elections) can be guessed from the offers of a set of gamblers.

(c) *Numerical example.* Carry out your method on the simple example below with $n = 5$ participants, $m = 5$ possible outcomes, and participant offers

Participant i	p_i	q_i	S_i
1	0.50	10	{1,2}
2	0.60	5	{4}
3	0.60	5	{1,4,5}
4	0.60	20	{2,5}
5	0.20	10	{3}

Compare the optimal worst-case house profit with the worst-case house profit, if all offers were accepted (*i.e.*, $x_i = q_i$). Find the imputed probabilities.

2. *Planning production with uncertain demand.* You must order (nonnegative) amounts $r_1, \ldots, r_m$ of raw materials, which are needed to manufacture (nonnegative) quantities $q_1, \ldots, q_n$ of n different products. To manufacture one unit of product j requires at least A_{ij} units of raw material i, so we must have $r \succeq Aq$. (We will assume that A_{ij} are nonnegative.) The per-unit cost of the raw materials is given by $c \in \mathbf{R}^m_+$, so the total raw material cost is $c^T r$.

   The (nonnegative) demand for product j is denoted d_j; the number of units of product j sold is $s_j = \min\{q_j, d_j\}$. (When $q_j > d_j$, $q_j - d_j$ is the amount of product j produced, but not sold; when $d_j > q_j$, $d_j - q_j$ is the amount of unmet demand.) The revenue from selling the products is $p^T s$, where $p \in \mathbf{R}^n_+$ is the vector of product prices. The profit is $p^T s - c^T r$. (Both d and q are real vectors; their entries need not be integers.)

   You are given A, c, and p. The product demand, however, is not known. Instead, a set of K possible demand vectors, $d^{(1)}, \ldots, d^{(K)}$, with associated probabilities $\pi_1, \ldots, \pi_K$, is given. (These satisfy $\mathbf{1}^T \pi = 1$, $\pi \succeq 0$.)

   You will explore two different optimization problems that arise in choosing r and q (the variables).

   I. Choose r and q ahead of time. You must choose r and q, knowing only the data listed above. (In other words, you must order the raw materials, and commit to producing the chosen quantities of products, before you know the product demand.) The objective is to maximize the expected profit.

   II. Choose r ahead of time, and q after d is known. You must choose r, knowing only the data listed above. Some time after you have chosen r, the demand will become known to you. This means that you will find out which of the K demand vectors is the true demand. Once you know this, you must choose the quantities to be manufactured. (In other words, you must order the raw materials before the product demand is known; but you can choose the mix of products to manufacture after you have learned the true product demand.) The objective is to maximize the expected profit.

   (a) Explain how to formulate each of these problems as a convex optimization problem. Clearly state what the variables are in the problem, what the constraints are, and describe the roles of any auxiliary variables or constraints you introduce.

   (b) Carry out the methods from part (a) on the problem instance with numerical data given in `planning_data.m`. This file will define `A`, `D`, `K`, `c`, `m`, `n`, `p` and `pi`. The K columns of D are the possible demand vectors. For both of the problems described above, give the optimal value of r, and the expected profit.

6.079 / 6.975 Introduction to Convex Optimization
Fall 2009