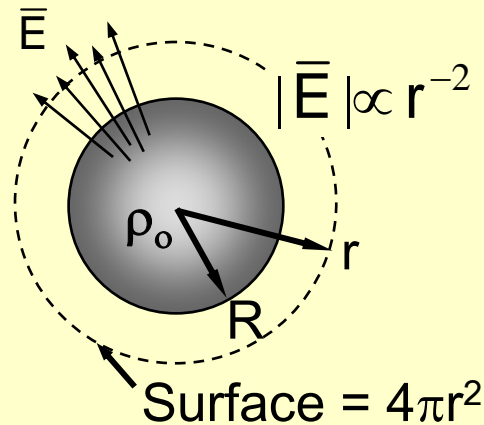


Estimating Static Fields

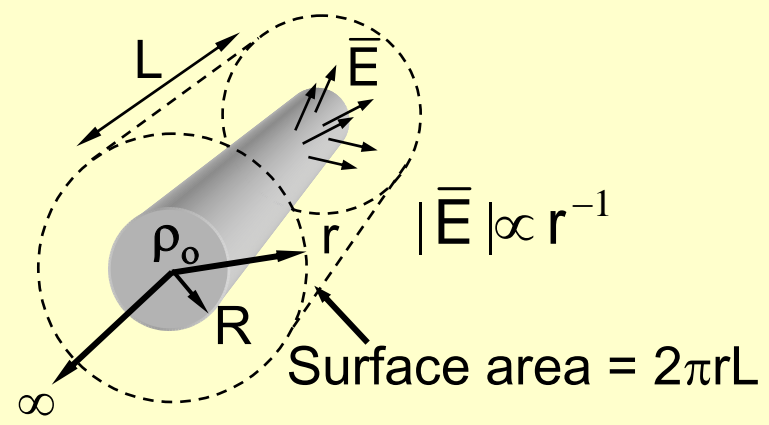
Examples
of point
and line
field
emission

Images of lightning rod, power line corona, and 0.1 micron integrated circuits removed due to copyright restrictions.

$$\Phi_{ab} = \int_a^b \vec{E} \cdot d\vec{z} \propto r^{-1} \text{ (spherical case), or } \propto \ln r \text{ (cylindrical case)}$$



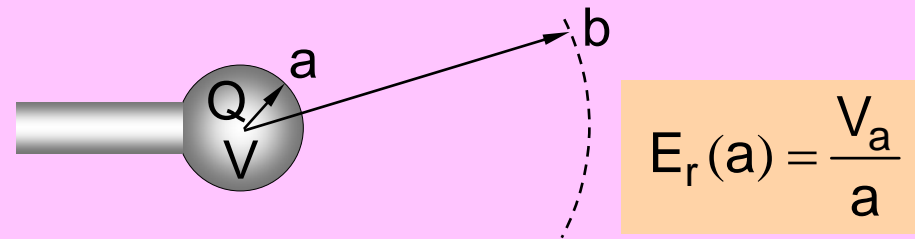
Spherical geometry



Cylindrical geometry

Simple Static Example

Spherical breakdown



$$Q = \int_A \bar{D} \cdot \hat{n} \, da = 4\pi r^2 \epsilon E_r(r) \Rightarrow E_r(r) = \frac{Q}{4\pi\epsilon r^2} = \frac{V_a 4\pi\epsilon a}{4\pi\epsilon r^2} = \frac{V_a a}{r^2}$$

$$V_a = \int_a^b E_r(r) dr = \frac{Q}{4\pi\epsilon} \int_a^b \frac{dr}{r^2} = -\frac{Q}{4\pi\epsilon} \left(\frac{1}{b} - \frac{1}{a} \right) \cong \frac{Q}{4\pi\epsilon a} \quad (b \gg a)$$

$\Rightarrow Q = V_a 4\pi\epsilon a$
 \Downarrow
 $V(r) = \frac{Q}{4\pi\epsilon r}$

Lightning rod: $a \cong 1 \text{ mm}, V \gtrsim 10^4 \text{ volts} \Rightarrow 10^7 \text{ V/m breakdown/corona}$

Power line: $a \cong 1 \text{ cm}, V \gtrsim 10^5 \text{ volts} \Rightarrow 10^7 \text{ V/m breakdown/corona}$

Integrated Circuit: $a \cong 0.1 \mu\text{m}, V = 1 \text{ volt} \Rightarrow 10^7 \text{ V/m breakdown/corona}$

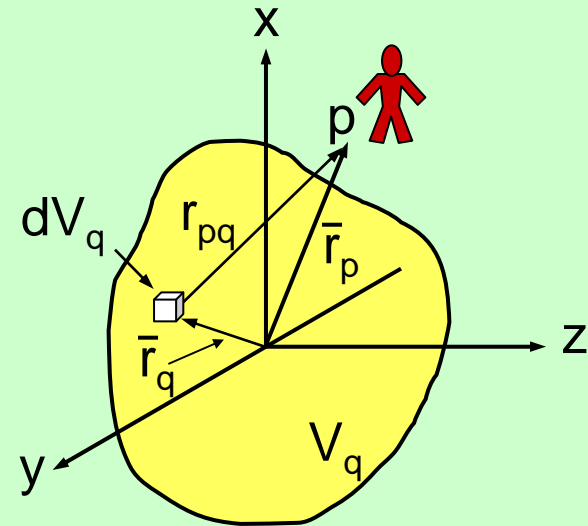
SOLVING FOR STATIC FIELDS

Approaches when given ρ, \bar{J} :

Use $\Phi_p = \int_{V_q} \frac{\rho_q}{4\pi\epsilon r_{pq}} dv, \quad \bar{E} = -\nabla\Phi$

$\bar{E}_p = \int_{V_q} \hat{r}_{pq} \frac{\rho_q}{4\pi\epsilon r_{pq}^2} dv$ Superposition integrals

$\bar{H}_p = \int_{V_q} \frac{\bar{J} \times (\bar{r}_p - \bar{r}_q)}{4\pi |\bar{r}_p - \bar{r}_q|^2} dv_q$ Biot-Savart Law



Approaches when given Φ, Ψ on boundaries:

Use Laplace's Equation. Derivation:

Electric: $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = 0$ (statics) $\Rightarrow \bar{E} = -\nabla\Phi. \quad \rho = 0 \Rightarrow \nabla \cdot \bar{E} = 0 \Rightarrow \nabla^2\Phi = 0$

Magnetic: $\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} = 0$ (statics; $\bar{J} = 0$) $\Rightarrow \bar{H} = -\nabla\Psi. \quad \nabla \cdot \bar{H} = 0 \Rightarrow \nabla^2\Psi = 0$

SEPARATION OF VARIABLES

Static charge-free regions obey Laplace's equation:

Electric potential $\nabla^2\Phi(\vec{r}) = 0$ Magnetic potential $\nabla^2\Psi(\vec{r}) = 0$

Assume $\Phi(x,y) = X(x)Y(y) \Rightarrow$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = Y(y) \frac{d^2X}{dx^2} + X(x) \frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X} \frac{d^2X}{dx^2} = -\frac{1}{Y} \frac{d^2Y}{dy^2} = -k^2 \text{ "separation constant"}$$

Solutions:

$$\frac{d^2X}{dx^2} = -k^2X \Rightarrow X(x) = A \cos(kx) + B \sin(kx)$$

$$\frac{d^2Y}{dy^2} = k^2Y \Rightarrow Y(y) = C \cosh(ky) + D \sinh(ky)$$

or: $Y(y) = C'e^{ky} + D'e^{-ky}$ (equivalent to above)

} For $k^2 > 0$;
(swap x,y
if $k^2 < 0$)

SEPARATION OF VARIABLES

Solution to Laplace's equation when $k^2 = 0$:

$$\Phi(x,y) = X(x)Y(y) \Rightarrow \frac{1}{X} \frac{d^2X}{dx^2} = -\frac{1}{Y} \frac{d^2Y}{dy^2} = 0$$

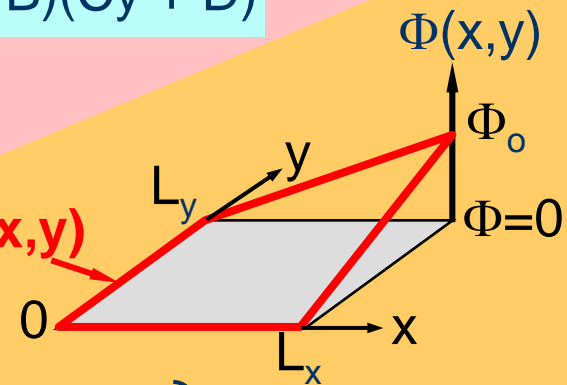
$$\frac{d^2X}{dx^2} = 0 \Rightarrow X(x) = Ax + B$$

$$\frac{d^2Y}{dy^2} = 0 \Rightarrow Y(y) = Cy + D$$

$$\Phi(x,y) = (Ax + B)(Cy + D)$$

Example, Cartesian coordinates:

Given $\Phi(x,y)$



$$\{\Phi = (Ax+B)(Cy+D) = 0 \text{ at } x=0 \text{ for all } y\} \Rightarrow B = 0$$

$$\{\Phi = (Ax+B)(Cy+D) = 0 \text{ at } y=0 \text{ for all } x\} \Rightarrow D = 0$$

$$\{\Phi = \Phi_0 \text{ at } x=L_x, y=L_y\} \Rightarrow AC = \Phi_0/L_xL_y$$

$$\Phi = xy\Phi_0/L_xL_y$$

Φ is matched at all 4 boundaries

CIRCULAR COORDINATES

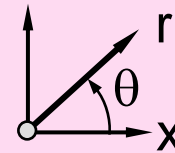
Separation of Laplace's equation:

Only in cartesian, cylindrical, spherical, and elliptical coordinates

Circular coordinates:

$$\nabla^2 \Phi(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0$$

$$\Phi = R(r)\Theta(\theta) \Rightarrow \frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = -\frac{1}{\Theta} \left(\frac{d^2 \Theta}{d\theta^2} \right) = m^2$$



Solutions (pick the one matching boundary condition):

$$\Phi(r, \theta) = (A + B\theta)(C + D \ln r) \quad \text{for } m^2 = 0$$

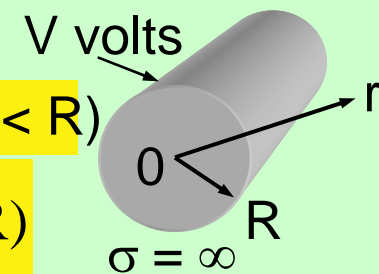
$$\Phi(r, \theta) = (A \sin m\theta + B \cos m\theta)(Cr^m + Dr^{-m}) \quad \text{for } m^2 > 0$$

$$\Phi(r, \theta) = [A \sinh m\theta + B \cosh m\theta][C \cos(m \ln r) + D \sin(m \ln r)] \quad \text{for } m^2 < 0$$

Example – conducting cylinder ($m = 0$):

$$\Phi(r, \theta) = C + D \ln r = V[2 - (\ln r / \ln R)] \quad (r \geq R); \quad \Phi(r, \theta) = V \quad (r < R)$$

$$\bar{E}(r) = -\nabla \Phi = -\left(\hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \right) \Phi = \frac{V}{r \ln R} \quad [\text{V/m}] \quad (r > R)$$



INHOMOGENEOUS MATERIALS

Governing Equations:

$$\bar{J} = \sigma \bar{E} \quad \bar{D} = \epsilon \bar{E} = \epsilon_0 \bar{E} + \bar{P} \quad \nabla \cdot \bar{D} = \rho_f \quad \nabla \cdot \bar{P} = -\rho_p$$

Non-uniform Conductivity $\sigma(x)$ (e.g., doping gradients in pn junctions):

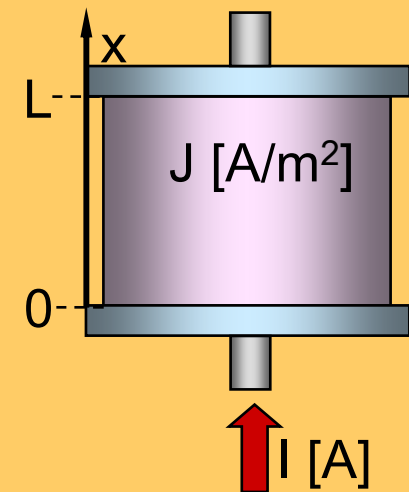
Assume $\sigma = \frac{\sigma_0}{1 + \frac{x}{L}}$ [S/m]

Note: Non-uniform conductors have free charge density ρ_f and polarization charge density ρ_p throughout.

$$\bar{E} = \frac{\bar{J}}{\sigma} = \hat{x} \frac{J_0(1 + \frac{x}{L})}{\sigma_0} \text{ [V/m]}$$

$$\rho_f = \nabla \cdot \bar{D} = \nabla \cdot \epsilon \bar{E} = \epsilon \frac{d}{dx} \left[\frac{J_0(1 + \frac{x}{L})}{\sigma_0} \right] = \frac{\epsilon J_0}{\sigma_0 L} \text{ [C/m}^3\text{]}$$

$$\rho_p = -\nabla \cdot \bar{P} = -\nabla \cdot (\epsilon - \epsilon_0) \bar{E} = -(\epsilon - \epsilon_0) \frac{d}{dx} \left[\frac{J_0(1 + \frac{x}{L})}{\sigma_0} \right] = -\frac{(\epsilon - \epsilon_0) J_0}{\sigma_0 L} \text{ [C/m}^3\text{]}$$



INHOMOGENEOUS PERMITTIVITY

Governing Equations:

$$\bar{J} = \sigma \bar{E} \quad \bar{D} = \epsilon \bar{E} = \epsilon_0 \bar{E} + \bar{P} \quad \nabla \cdot \bar{D} = \rho_f \quad \nabla \cdot \bar{P} = -\rho_p$$

Example, Non-uniform Permittivity $\epsilon(x)$:

Assume: $\epsilon = \epsilon_0 \left(1 + \frac{x}{L}\right)$ [F/m]

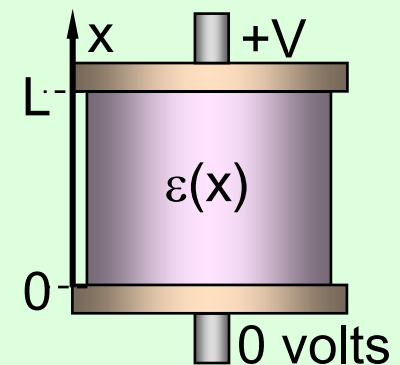
$$\bar{D} = \epsilon \bar{E} = f(x) \Rightarrow E = \frac{D_0}{\epsilon} = \frac{D_0}{\epsilon_0 \left(1 + \frac{x}{L}\right)} = \frac{E_0}{\left(1 + \frac{x}{L}\right)}$$

$$V = \int_0^L E_x dx = \int_0^L \frac{E_0}{1 + \frac{x}{L}} dx = E_0 L \ln 2 \text{ [V]} \Rightarrow E_0 = \frac{V}{L \ln 2} \text{ [V/m]}$$

Therefore: $\bar{E} = \hat{x} \frac{V}{L \ln 2 \left(1 + \frac{x}{L}\right)}$ [V/m] Note: Non-uniform E(x)

$$\rho_p = -\nabla \cdot \bar{P} = -\nabla \cdot (\bar{D} - \epsilon_0 \bar{E}) = \epsilon_0 \nabla \cdot \bar{E} = (V/L \ln 2) \frac{d}{dx} \left(1 + \frac{x}{L}\right)^{-1} = \frac{-V}{L^2 \ln 2 \left(1 + \frac{x}{L}\right)^2}$$

Non-uniform dielectrics have polarization charge density ρ_p throughout.



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