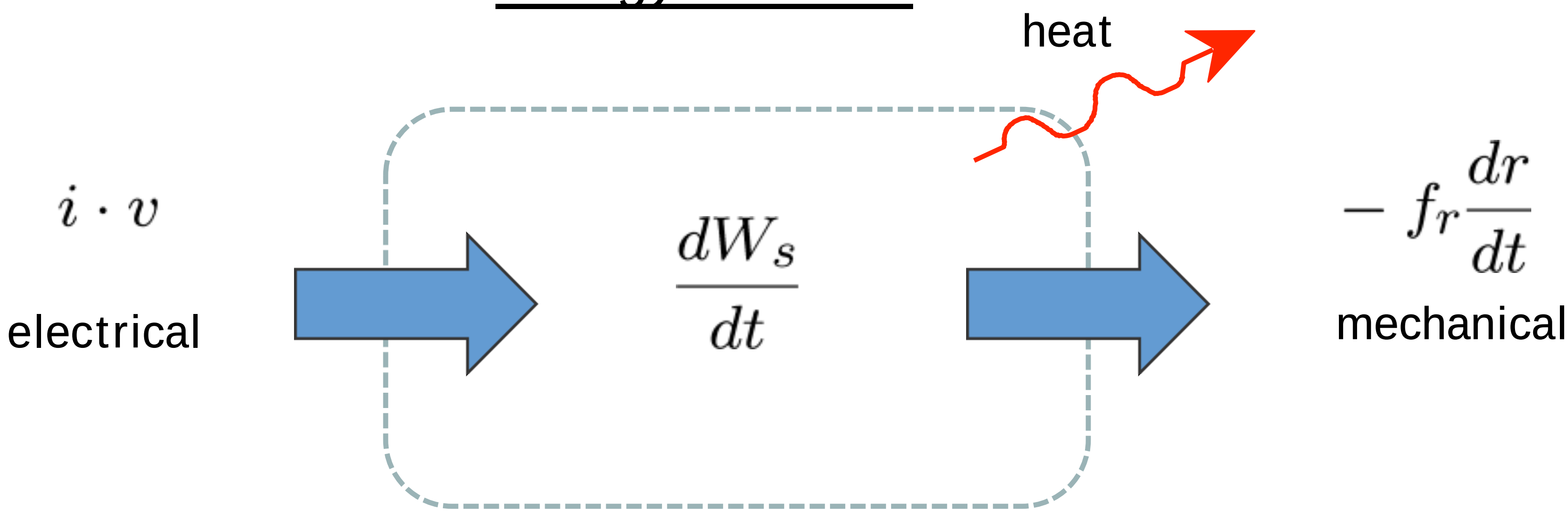
Stored Energy and Simple Actuators

<u>Outline</u>

Magnetic-Field Actuators
Electric-Field Actuator

Energy Balance



$$\frac{dW_s(\lambda\,,r)}{dt} = \frac{\partial W_s}{\partial \lambda}\frac{d\lambda}{dt} + \frac{\partial W_s}{\partial r}\frac{dr}{dt}$$

neglect heat

For magnetostatic system, $d\lambda/dt=0$ **no electrical power** flow ...

$$\frac{dW_s}{dt} = -f_r\frac{dr}{dt} \quad \Longrightarrow \quad f_r = -\frac{\partial W_s}{\partial r}\Big|_\lambda$$

Radial Force on the Solenoid

(at a constant flux linkage, λ)

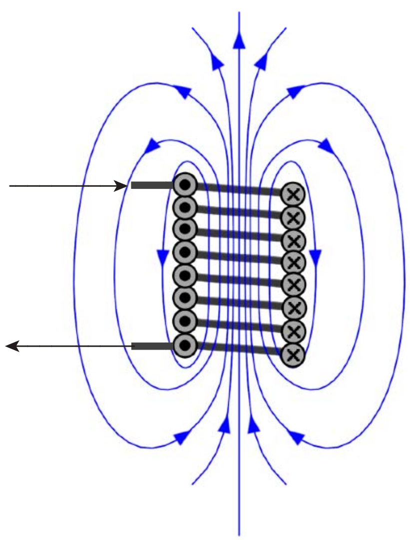
If we can find the stored energy, we can immediately compute the force...

...lets take all the things we know to put this together...

$$f_r = -\frac{\partial W_s}{\partial r}|_\lambda \qquad W_s(\Phi, r) = \frac{1}{2}\frac{\lambda^2}{L} \qquad L(r) = \frac{\mu_o N^2 \pi r^2}{h}$$

true for any single coil

true for long solenoid



$$f_r = -\frac{\partial W_s}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\lambda^2}{L}\right) = \frac{1}{2}\frac{\lambda^2}{L^2}\frac{\partial L}{\partial r} = \frac{1}{2}i^2\frac{\partial L}{\partial r}$$

$$f_r = \frac{\mu_o N^2 i^2}{h} \cdot \pi r$$

Stored Energy in Inductors

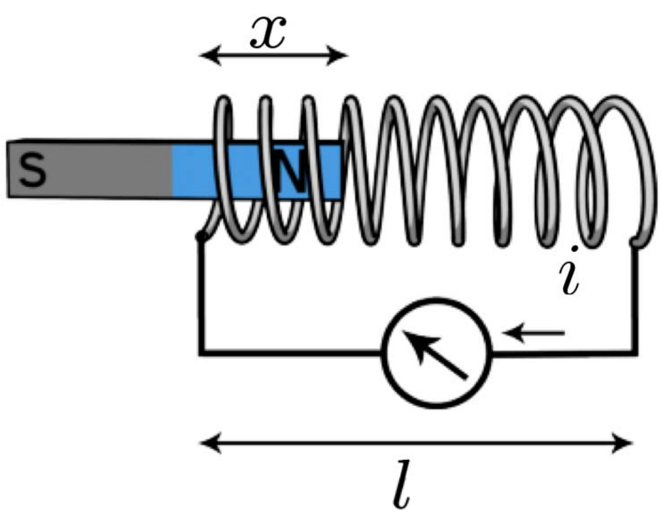
In the absence of mechanical displacement, ...

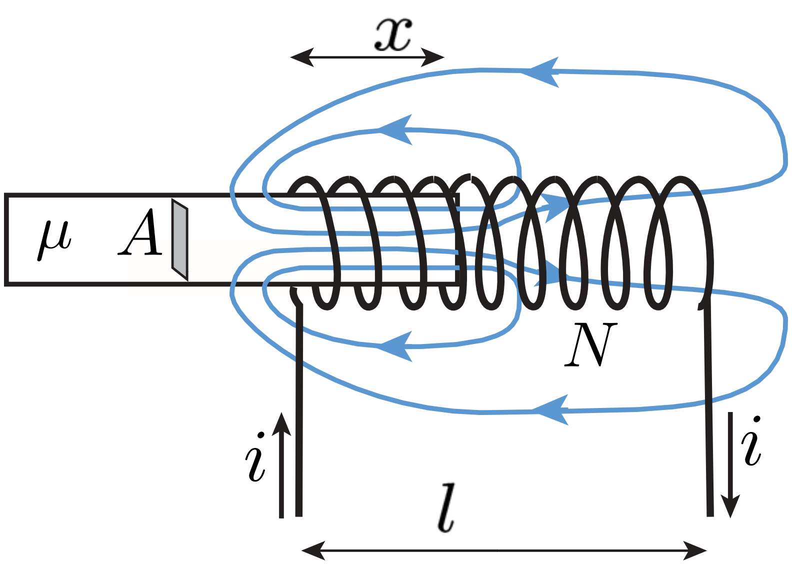
$$W_s = \int P_{elec}\, dt = \int iv \ \ dt = \int i \frac{d\lambda}{dt} dt = \int i(\lambda) d\lambda$$

For a linear inductor:

$$i(\lambda) = \frac{\lambda}{L} \quad \Longrightarrow \quad W_s = \int_0^{\lambda} \frac{\lambda'}{L} d\lambda' = \frac{\lambda^2}{2L}$$

Linear Machines: Solenoid Actuator





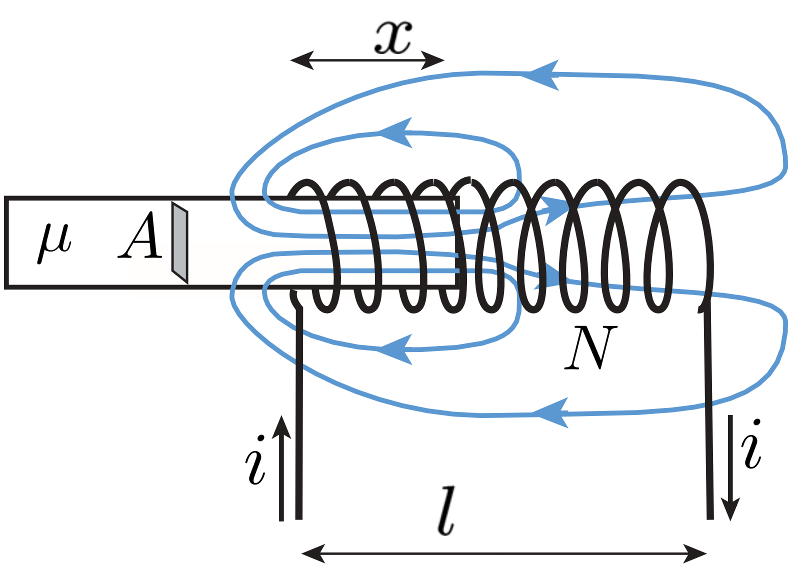
Assuming that H outside the solenoid is negligible

$H = \dfrac{Ni}{l}$ inside the solenoid

$B_{core} \approx \mu H$ and $B_{air} \approx \mu_o H$

Example: Solenoid with a Core

- Inductance: $\lambda = B_{core}ANx/l + B_{air}AN(l-x)/l \equiv L(x)i$
- Force: $f = \frac{1}{2}i^2\frac{dL(x)}{dx} = \frac{1}{2}i^2N^2A\frac{(\mu-\mu_o)}{d^2}$



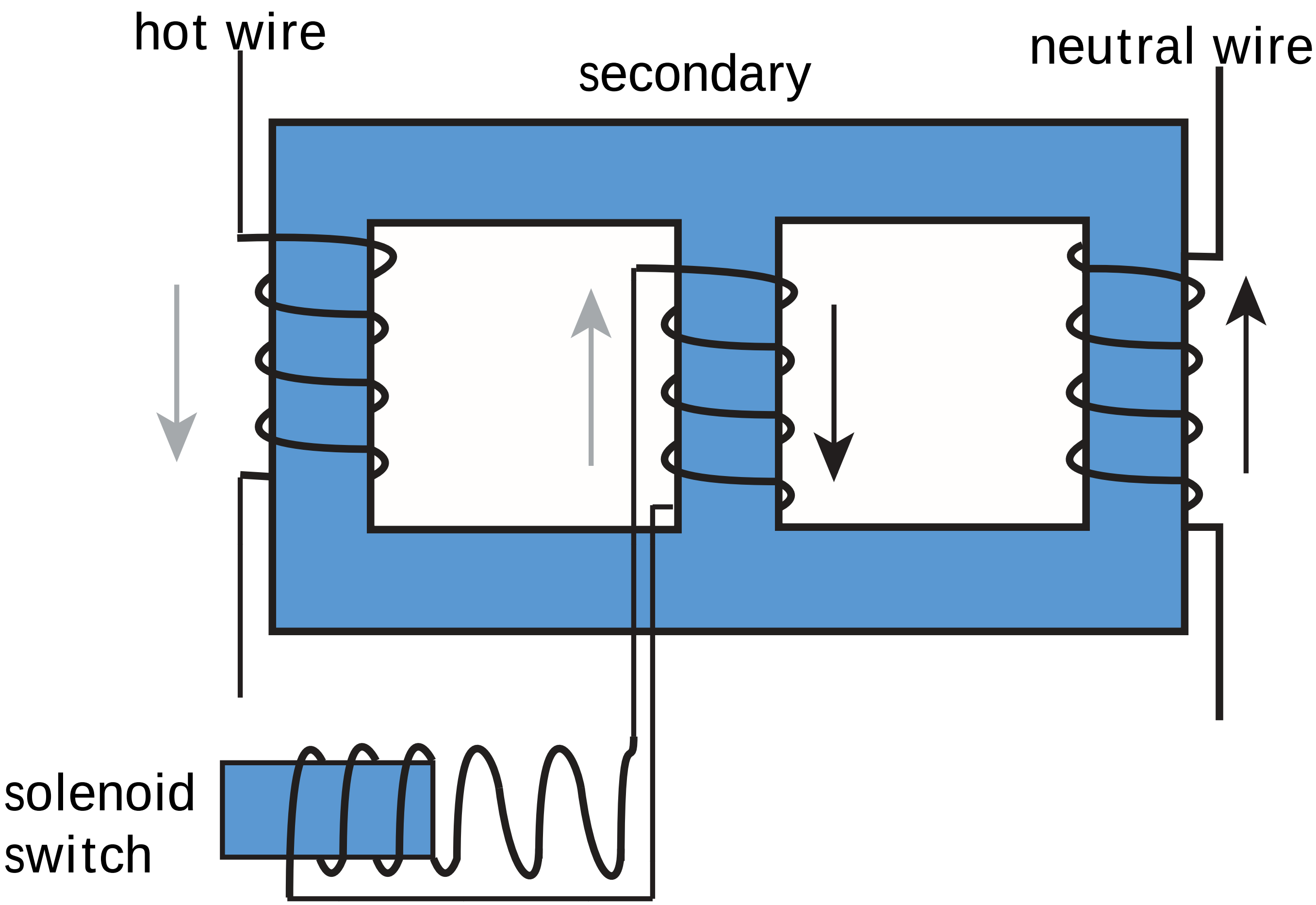
$$W_S(\lambda,x) = \frac{\lambda^2}{2L(x)}$$

$$L(x) = \frac{\mu_0 A}{l-x}\left[N\frac{l-x}{l}\right]^2 + \frac{\mu A}{x}\left[N\frac{x}{l}\right]^2$$

$$f = -\left(\frac{\partial W_S}{\partial x}\right)_\lambda = \frac{\lambda^2 l^2(\mu-\mu_0)}{2N^2A\left[\mu_0(l-x)+\mu x\right]^2}$$

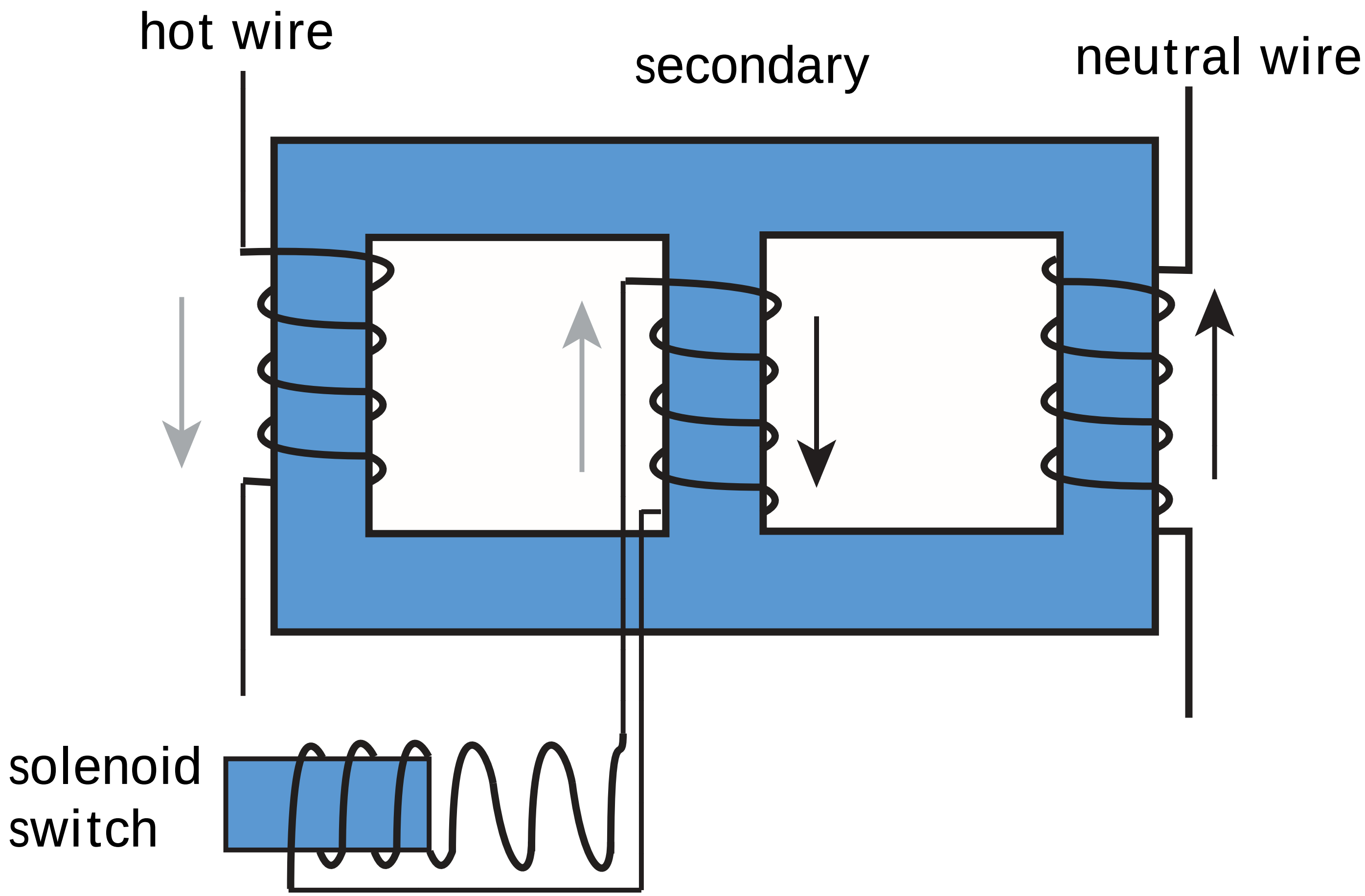
THE CORE IS PULLED INTO THE SOLENOID
(FORCE ACTS TO INCREASE L)

Example: Differential Transformer



If the current in the hot wire is the same as the current in the neutral wire, the induced current in the secondary is zero.

Example: Differential Transformer

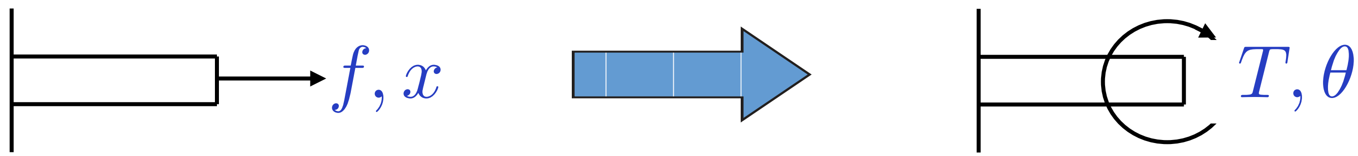


If some current is lost,

current in the secondary opens the solenoid switch.

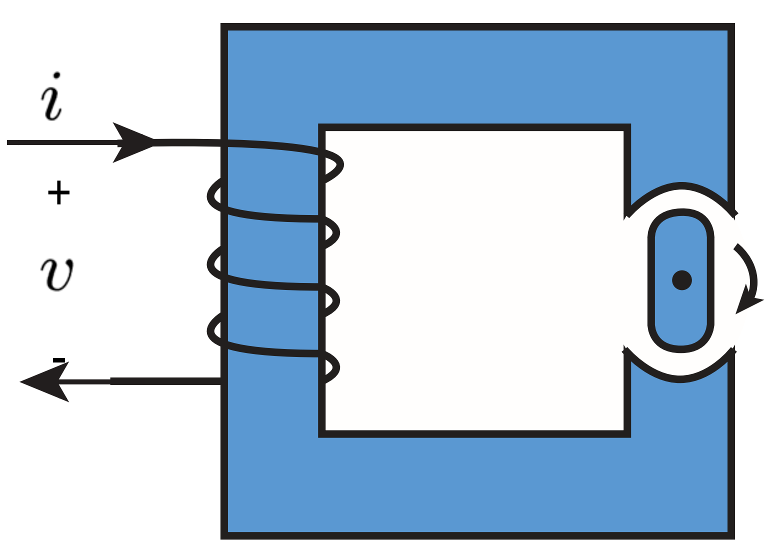
Rotational Mechanics ... Motors & Generators

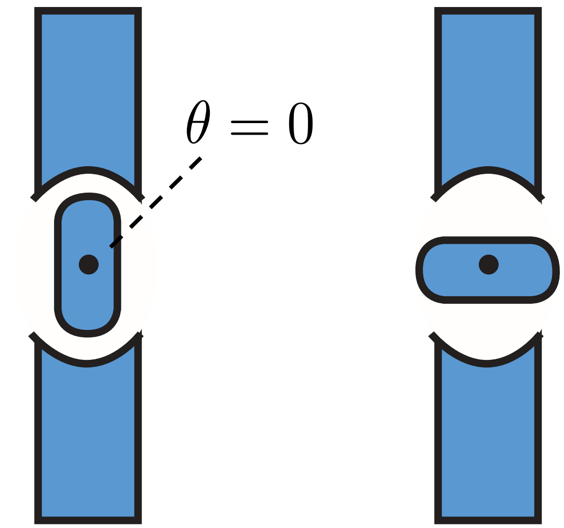
The translational variables *x* and *f* become
the rotational variables *θ* and *T*.
All else remains the same.



$$f = \frac{1}{2} i^2 \frac{dL(x)}{dx} \qquad\qquad T = \frac{1}{2} i^2 \frac{dL(\theta)}{d\theta}$$

Rotating Machine: Variable-Reluctance Motor





Max inductance   Min inductance

$$L(\theta) = L_0 + L_2\, sin(2\theta)$$

$$T = \frac{1}{2}\, i^2\, \frac{dL(\theta)}{d\theta} = L_2\, i^2 cos(2\theta)$$

THE ROTATING CORE IS PULLED INTO ALIGNMENT WITH THE STATIONARY CORE (AGAIN, THE FORCE ACTS TO MAXIMIZE INDUCTANCE)

Image by Steve Johnson
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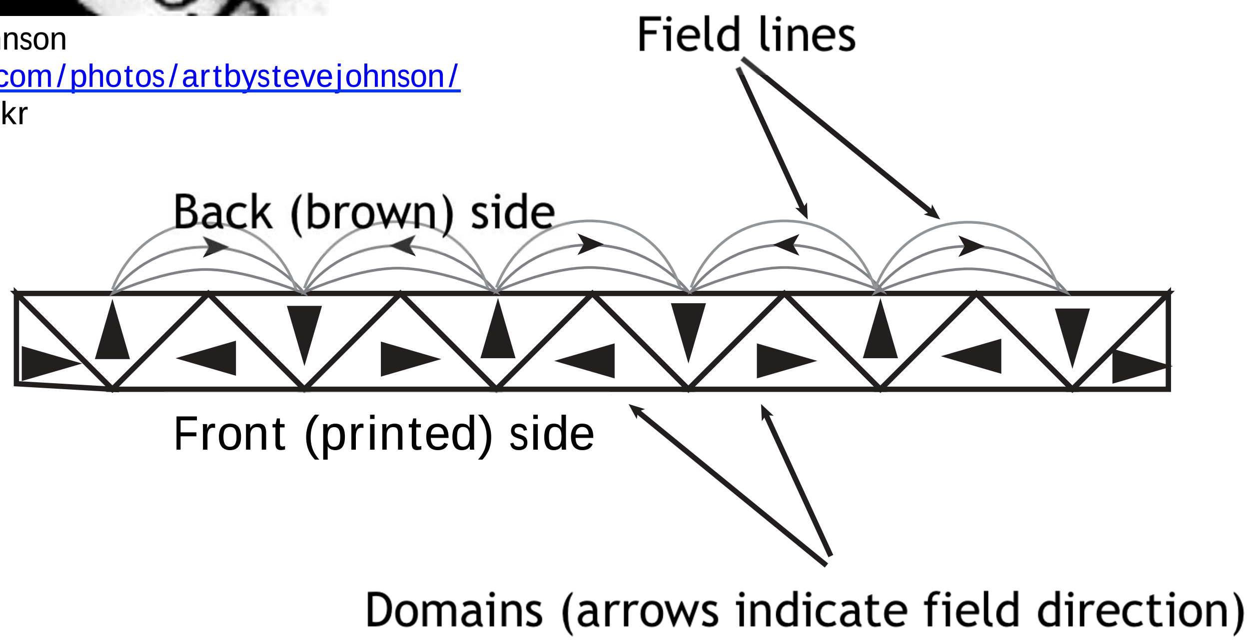
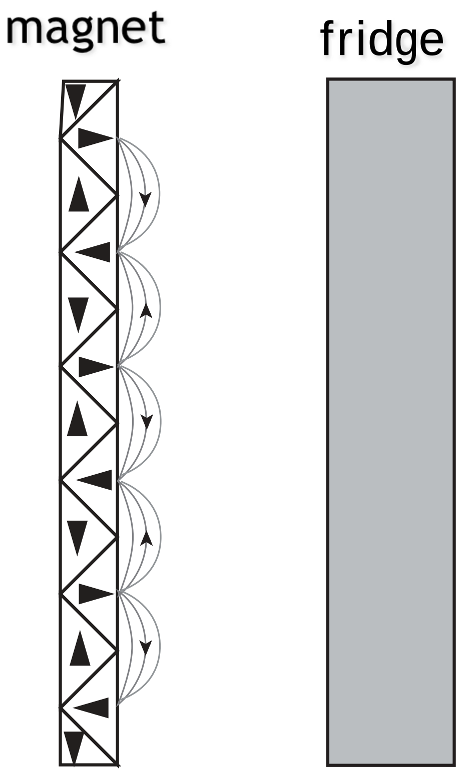
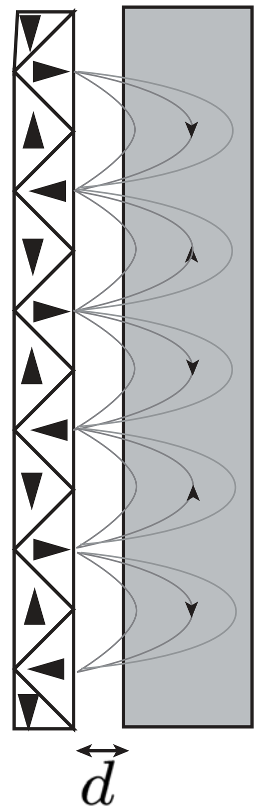
Magnetic Poetry

magnetic strontium ferrite, $SrFe_{12}O_{19}$, particles dispersed in the elastomer Hypalon



Magnetic Poetry





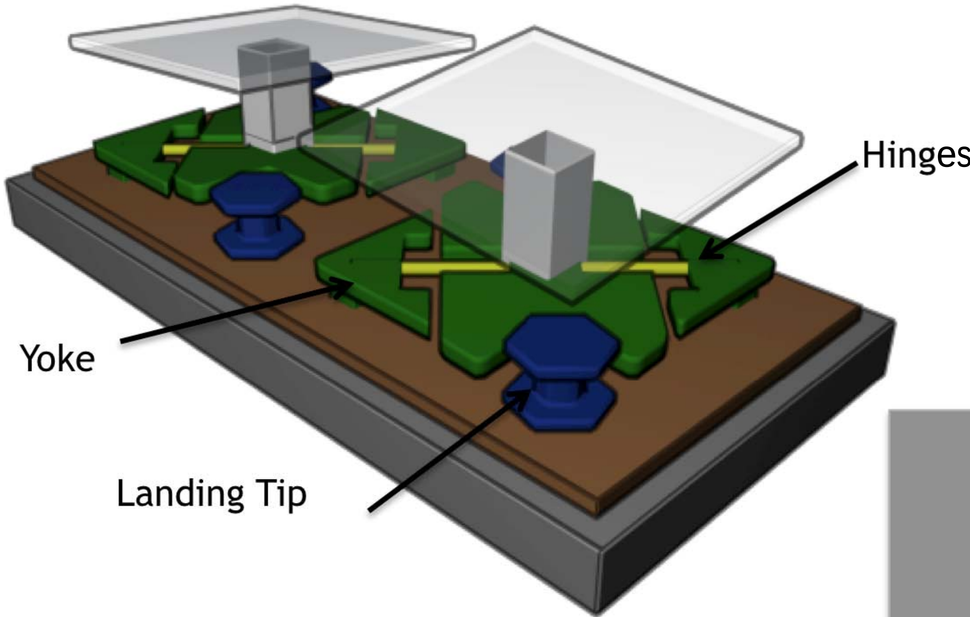
$$f_r = -\frac{\partial W_s}{\partial d}$$

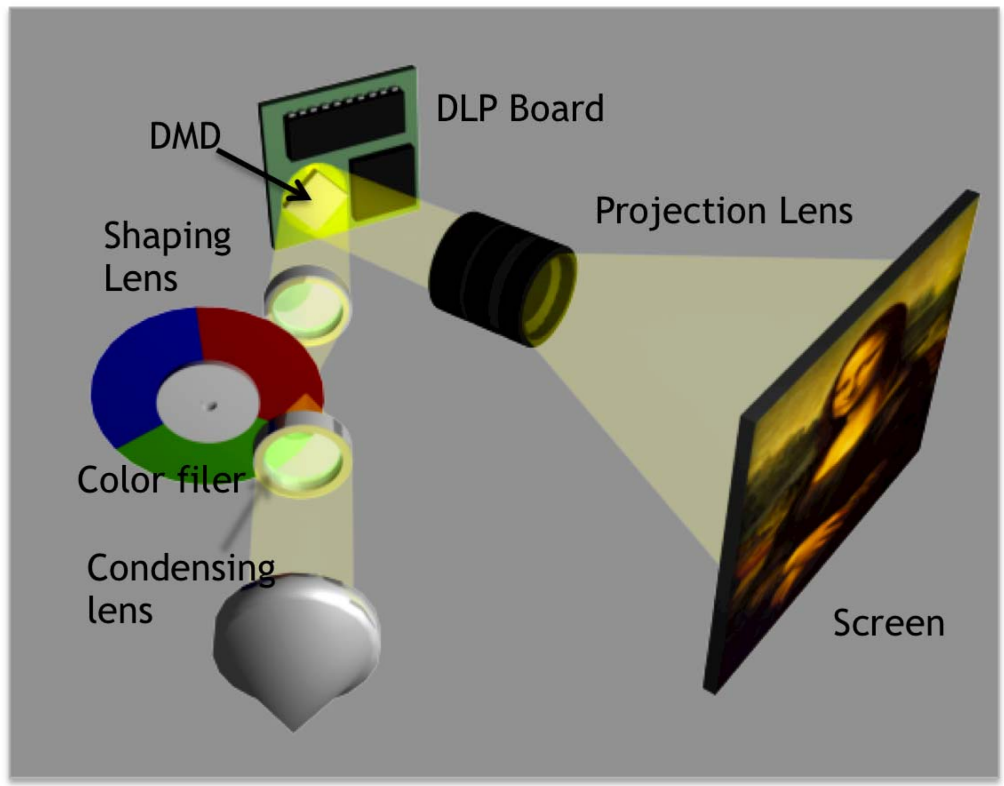
$$\frac{W_s}{V} = \frac{1}{2}\mu_o H_{gap}^2$$

$$W_s = (Ad)\frac{1}{2}\frac{B_{gap}^2}{\mu_o}$$

$$\underbrace{\frac{f}{A}}_{\text{PRESSURE}} = \frac{B_{gap}^2}{2\mu_o}$$

Ubiquitous Electrostatic Motors





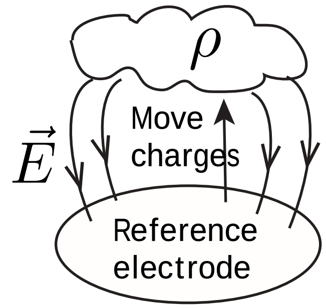
Energy Stored in Electric Fields

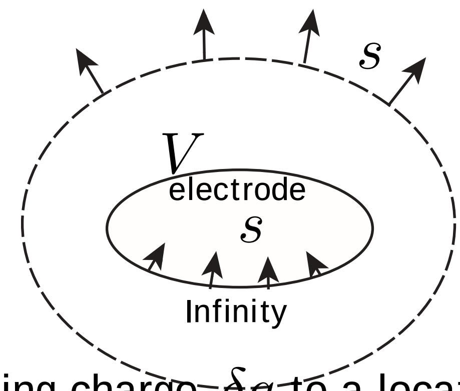
1. Begin with a neutral reference conductor, the charge reservoir. Its potential is zero, by definition.

2. Move charges from the reference conductor into free space, thereby creating an electric field and doing work in the process. The work is stored as potential energy in the electric fields.

3. Account for all the work done, and thereby derive the energy stored in the electric fields.

4. The argument directly extends to systems with multiple conductors (and dielectrics).





- The work done by moving charge δq to a location with potential U is $U\,\delta q$. More generally, the work done to make an incremental charge change to a charge density is

$$\delta w = \iiint_V U\,\delta\rho\,dV$$

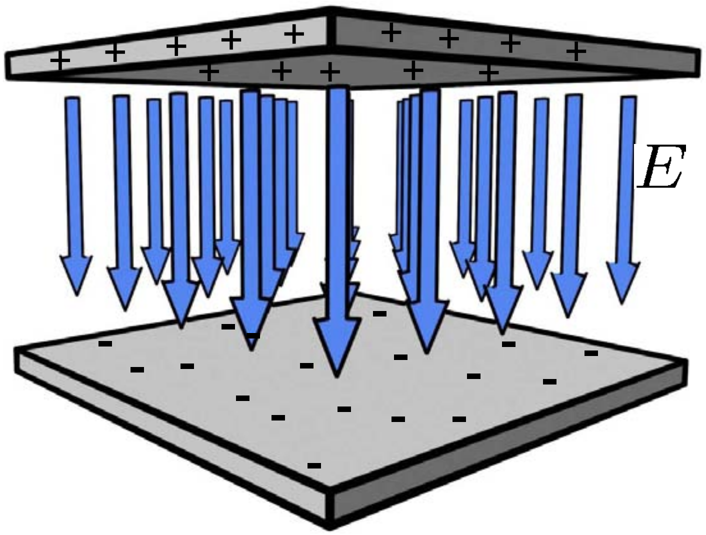
- Gauss' Law $\Rightarrow$ $\quad \delta\rho = \nabla\cdot\delta\epsilon_o\vec{E} \quad \Rightarrow$

$$\begin{aligned}
\delta w &= \int\int\int_V U\nabla\cdot\delta\epsilon_o\vec{E}dV \\
&= \int\int\int_V [\nabla\cdot(U\delta\epsilon_o\vec{E}) - \delta\epsilon_o\vec{E}\cdot\nabla U]dV \\
&= \underbrace{\oint U\delta\epsilon_o\vec{E}\cdot d\vec{S}}_{\text{ZERO !}} + \int\int\int_V \vec{E}\cdot\delta\epsilon_o\vec{E}dV
\end{aligned}$$

ENERGY DENSITY [J/m³]

$$\boxed{\frac{\delta W}{\delta V} = \vec{E}\cdot\epsilon_o\delta\vec{E} \quad \Rightarrow \quad \frac{W}{Volume} = \frac{1}{2}\epsilon_o E^2}$$

First Attempt at Estimating Forces



$$+\sigma$$

$$q = Cv$$

$$-\sigma$$

$$\epsilon_o E A = \sigma A$$

$$E = \frac{\sigma}{\epsilon_o}$$

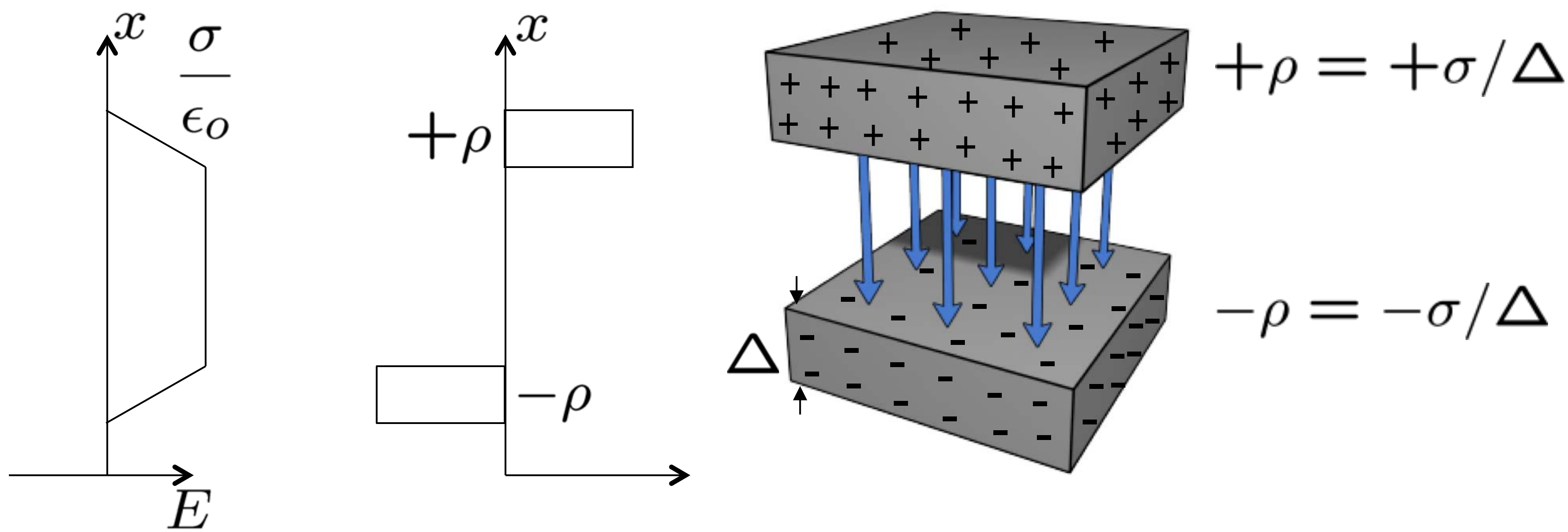
$$f = qE = \sigma A \frac{\sigma}{\epsilon_o}$$

$$= qE = \frac{\sigma^2 A}{\epsilon_o} = \epsilon_o E^2 A$$

... this analysis is wrong ! Wrong answer ! Why?

Second Try at Estimating Forces

If the capacitor plates have finite thickness,
most of the charge density doesn't see the peak field … E



$$f = \underbrace{\frac{1}{2}\frac{\sigma}{\epsilon_o}}_{\text{Avg. E}} \underbrace{\frac{\sigma}{\Delta}}_{\rho} \underbrace{\Delta A}_{V} = \frac{\sigma^2 A}{2\epsilon_o} = \frac{1}{2}\epsilon_o E^2 A$$

Third Try: Use the Energy Method

Relate Stored Energy to Force

Lets use chain rule...

$$\frac{dW_s(q,x)}{dt} = \frac{\partial W_s}{\partial q}\frac{dq}{dt} + \frac{\partial W_s}{\partial x}\frac{dx}{dt}$$

This looks familiar...

$$\begin{aligned}\frac{dW_s}{dt} &= i \cdot v - f\frac{dx}{dt} \\ &= C\frac{dv}{dt}v - f\frac{dx}{dt}\end{aligned}$$

Comparing similar terms suggests...

$$\boxed{f = -\frac{\partial W_s}{\partial x}}$$

Stored Energy of a Capacitor

$$P_{elec} = v \cdot i = v \cdot \frac{dq}{dt}$$

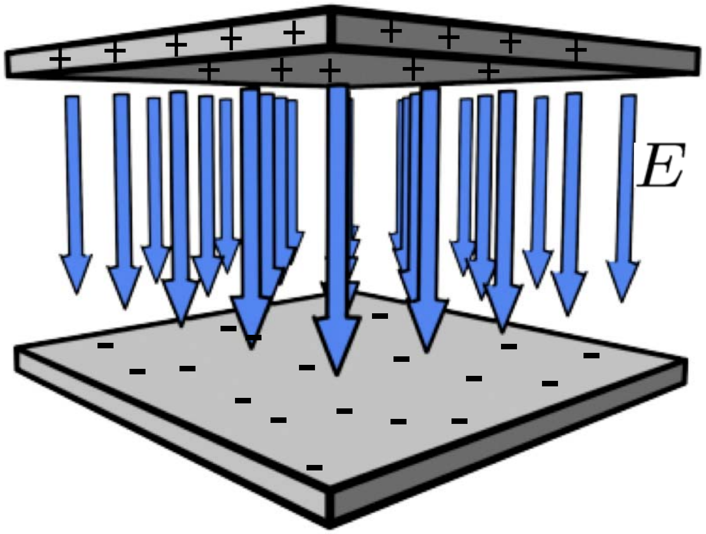
$$W_s = \int P_{elec}\, dt = \int v \frac{dq}{dt} dt = \int v\ dq$$

... where W_s is energy stored in the field of the capacitor at any instant in time

$$W_s = \boxed{}$$

$$f = \boxed{}$$

Third Try: Attractive Force Between Parallel Plates



$$f = \frac{v^2}{2}\frac{dC}{dx}$$

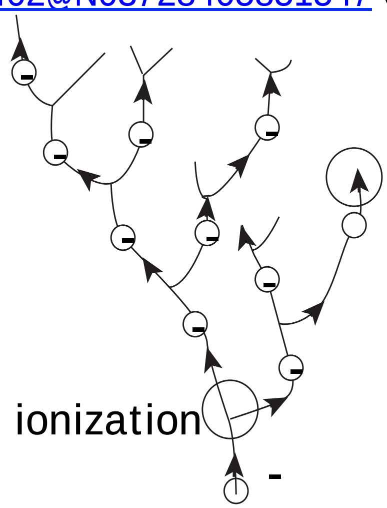
$$C(x) = \frac{\epsilon_o A}{x}$$

CAPACITOR PLATES ARE PULLED TOWARDS EACH OTHER (FORCE ACTS TO INCREASE C)

$$f = \boxed{}$$

$$f = \frac{1}{2}\epsilon_o E^2 A$$

Image by Bruce Guenter http://www.flickr.com/photos/10154402@N03/2840585154/ on flickr



Electron thermalizes due to collisions

How Strong is this Force ?

The maximum electric field strength is limited by the electrostatic breakdown

Typically...

$$E_{max} \approx 10^6 \ \frac{\text{V}}{\text{m}}$$

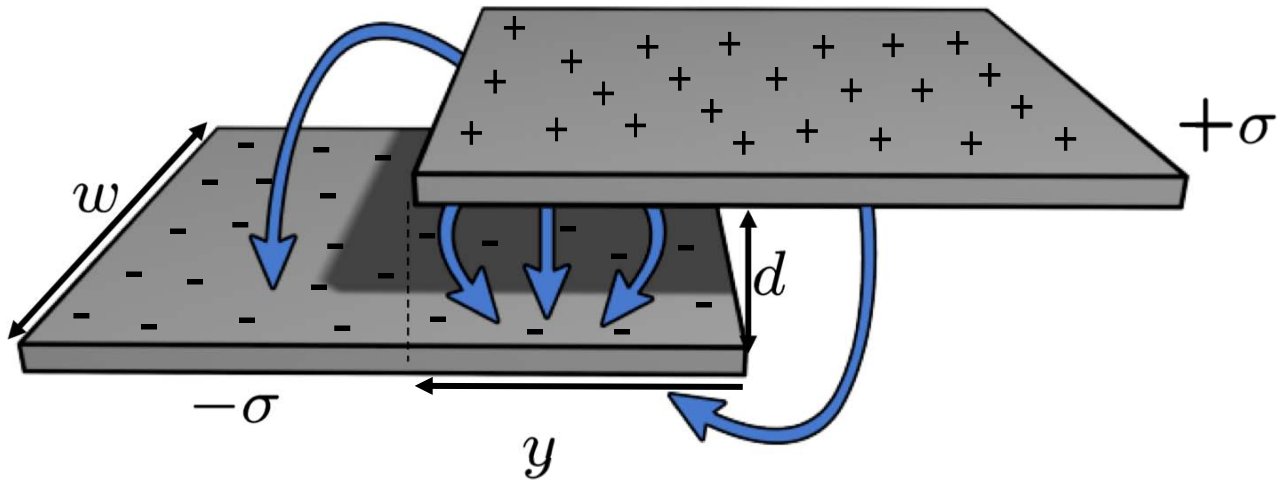
$$\frac{f}{A} = \frac{1}{2}\epsilon_o E^2 \approx \boxed{}$$

For small gaps...

$$E_{max} \approx 10^8 \ \frac{\text{V}}{\text{m}}$$

$$\frac{f}{A} = \frac{1}{2}\epsilon_o E^2 \approx \boxed{}$$

Linear Electrostatic Actuator



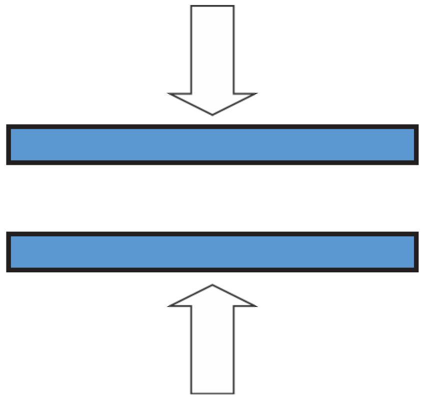
$$C(y) = \frac{\epsilon_o(yw)}{d}$$

$$f = \boxed{}$$

CONSTANT FORCE ALONG DIRECTION OF MOTION

(FORCE ACTS TO INCREASE C)

Gap Closing Electrostatic Actuators

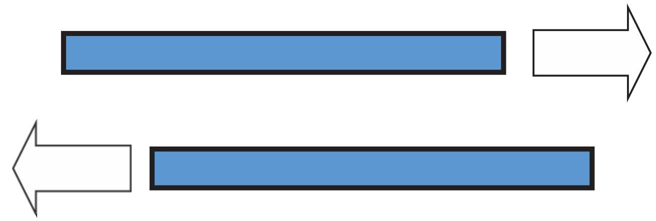


$$C(x) = \frac{\epsilon_o A}{x}$$

$$f = -\frac{v^2}{2}\frac{\epsilon_o A}{x^2}$$

Moderate force

$$\sim \frac{1}{x^2}$$



$$C(y) = \frac{\epsilon_o (yw)}{d}$$

$$f(y) = \frac{\epsilon_o}{2}\frac{Q^2}{C^2}\frac{w}{d}$$

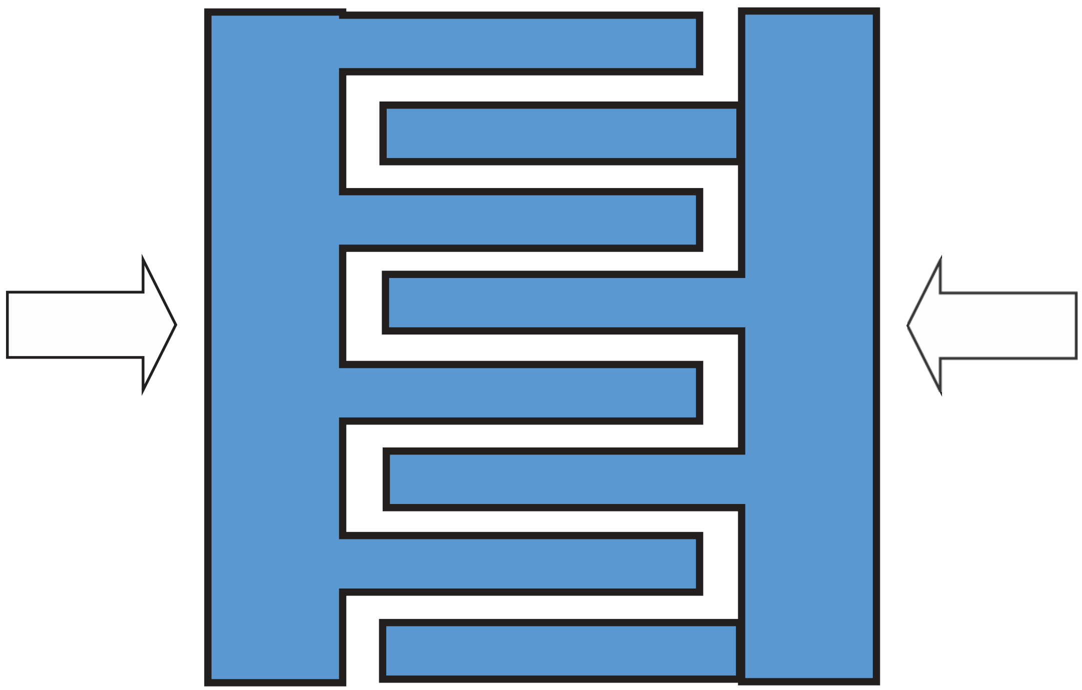
Weak force

$$\sim \frac{1}{d}$$

Example: Linear Comb Drive

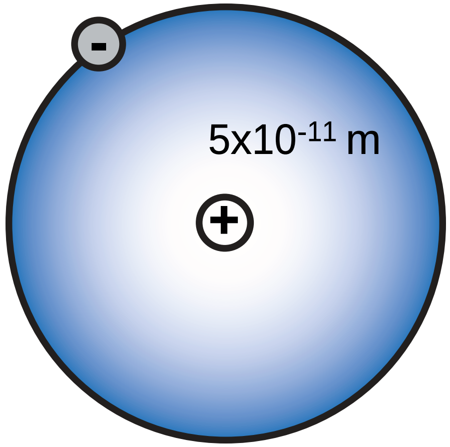
N-fold multiplication of force ...

$$f = N\epsilon_o \frac{v^2}{2}\frac{w}{d}$$



Modeling Atoms and Molecules as Capacitors that Store Energy

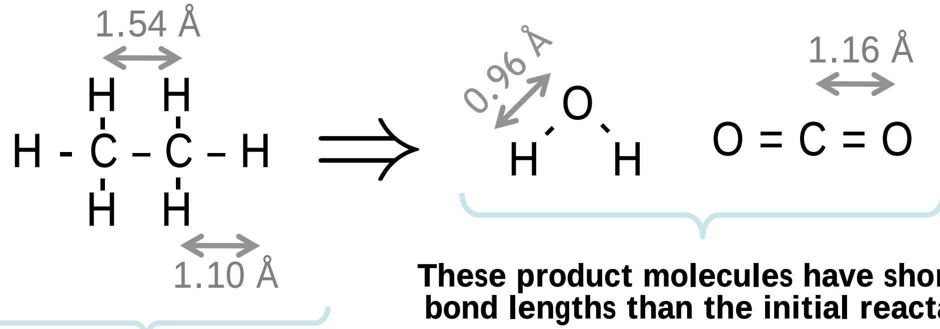
$$(r) = \frac{q}{4\pi\epsilon_o r}$$



Hydrogen atom

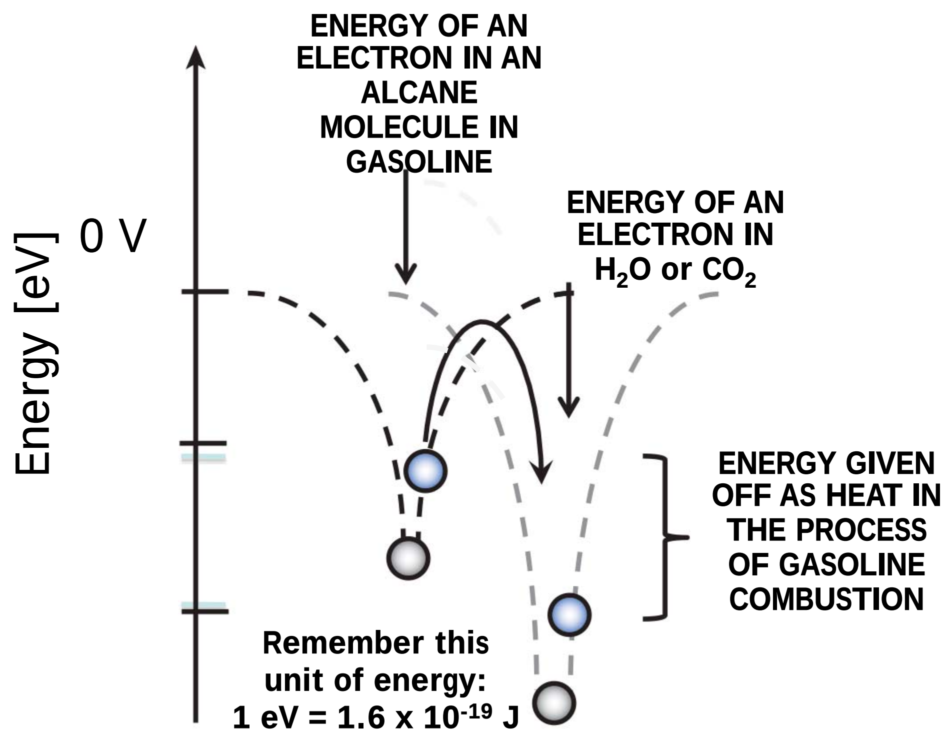
Hydrogen ground state energy is -13.6 eV

If the hydrogen radius was twice as long, what would be the ground state energy ?



Burn this molecule by reacting it with oxygen

These product molecules have shorter bond lengths than the initial reactant molecule, hence the charge in them sits closer together. This can be modeled as a higher capacitance. Since voltage V = Q/C = E • d is reduced, stored energy W/Volume = ½ $\varepsilon_o E^2$ is reduced in these molecules



Electrostatic vs Magnetostatic Actuators

MAGNETIC

$$\frac{f}{A} = \frac{W_S}{V} = \frac{1}{2}\mu H \cdot H$$

ELECTRIC

$$\frac{W_S}{V} = \frac{1}{2}\epsilon E \cdot E$$

	Max Field	$\frac{W_s}{V}$
Magnetic	$H_{max} \approx 1\ T$	400 kJ/m^3
Electric (Macro)	$E_{max} \approx 10^6\ \frac{\mathrm{V}}{\mathrm{m}}$	4.4 J/m^3
Electric (Micro)	$E_{max} \approx 10^8\ \frac{\mathrm{V}}{\mathrm{m}}$	44 kJ/m^3
Electric (Bio/Nano)	$E_{max} \approx 10^9\ \frac{\mathrm{V}}{\mathrm{m}}$	4.4 MJ/m^3
Gasoline		38 GJ/m^3

$$\mu_o = 4\pi \times 10^{-7}\,\mathrm{H/m}$$

$$\epsilon_o = 8.854 \times 10^{-12}\,\mathrm{F/m}$$

KEY TAKEAWAYS

Energy method for calculating Forces
calculated at constant flux linkage

$$f_r = -\frac{\partial W_s}{\partial r}$$

Radial force
for an inductor

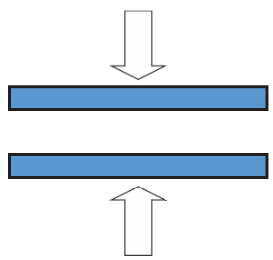
$$f_r = -\frac{\partial W_s}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\lambda^2}{L}\right) = \frac{1}{2}\frac{\lambda^2}{L^2}\frac{\partial L}{\partial r} = \frac{1}{2}i^2\frac{\partial L}{\partial r}$$

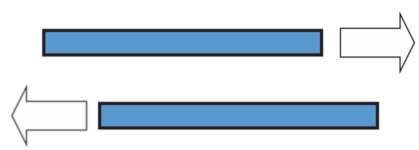
FORCE ACTS TO INCREASE INDUCTANCE, L

$$W_s = \int_0^{\lambda} \frac{\lambda'}{L}d\lambda' = \frac{\lambda^2}{2L}$$

FORCE ACTS TO INCREASE CAPACITANCE, C

$$C(x) = \frac{\epsilon_o A}{x}$$





$$C(y) = \frac{\epsilon_o(yw)}{d}$$

$$f = -\frac{v^2}{2}\frac{\epsilon_o A}{x^2}$$

$$f(y) = \frac{\epsilon_o}{2}\frac{Q^2}{C^2}\frac{w}{d}$$

Moderate force $\sim \frac{1}{x^2}$

Weak force $\sim \frac{1}{d}$

6.007 Electromagnetic Energy: From Motors to Lasers
Spring 2011