<u>Electrostatics</u>

(Free Space With Charges & Conductors)

Reading - Shen and Kong – Ch. 9

<u>Outline</u>

Maxwell's Equations (In Free Space)
Gauss' Law & Faraday's Law
Applications of Gauss' Law
Electrostatic Boundary Conditions
Electrostatic Energy Storage

Maxwell's Equations (in Free Space with Electric Charges present)

	DIFFERENTIAL FORM	INTEGRAL FORM
E-Gauss:	$\nabla \cdot \epsilon_o \vec{E} = \rho$	$\oint\!\!\oint_S \epsilon_o \vec{E} \cdot dS = \iiint_V \rho dV$
Faraday:	$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \mu_o \vec{H}$	$\oint_C \vec{E} \cdot d\vec{C} = \iint_S -\frac{\partial}{\partial t} \mu_o \vec{H} \cdot d\vec{S}$
H-Gauss:	$\nabla \cdot \mu_o \vec{H} = 0$	$\oint\!\!\oint_S \mu_o \vec{H} \cdot d\vec{S} = 0$
Ampere:	$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t} \epsilon_o \vec{E}$	$\oint_C \vec{H} \cdot d\vec{C} = \iint_S (\vec{J} + \frac{\partial}{\partial t} \mu_o \vec{H}) \cdot d\vec{S}$

Static arise when $\frac{\partial}{\partial t} \equiv 0$, and Maxwell's Equations split into **decoupled electrostatic** and **magnetostatic** eqns.
Electro-quasistatic and magneto-quasitatic systems arise when one (but not both) time derivative becomes important.

Note that the Differential and Integral forms of Maxwell's Equations are related through

Stoke's Theorem $\oint_C A \cdot dC = \iint_S \nabla \times A \cdot dS$ and Gauss' Theorem $\oint\!\!\oint_S A \cdot dS = \iiint_V \nabla \cdot A \, dV$

Charges and Currents

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\epsilon_o \vec{E}$$

$$\nabla \cdot \epsilon_o \vec{E} = \rho$$

$$\Rightarrow$$

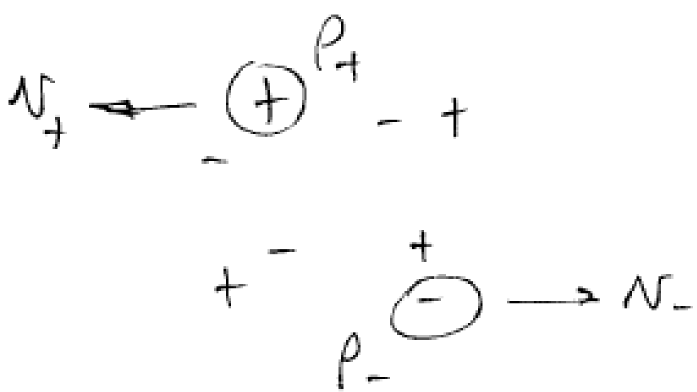
$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\oiint_S \vec{J} \cdot d\vec{S} + \iiint_V \frac{\partial \rho}{\partial t} dV = 0$$

Charge conservation and KCL for ideal nodes

There can be a nonzero charge density ρ in the absence of a current density J.

There can be a nonzero current density J in the absence of a charge density ρ.



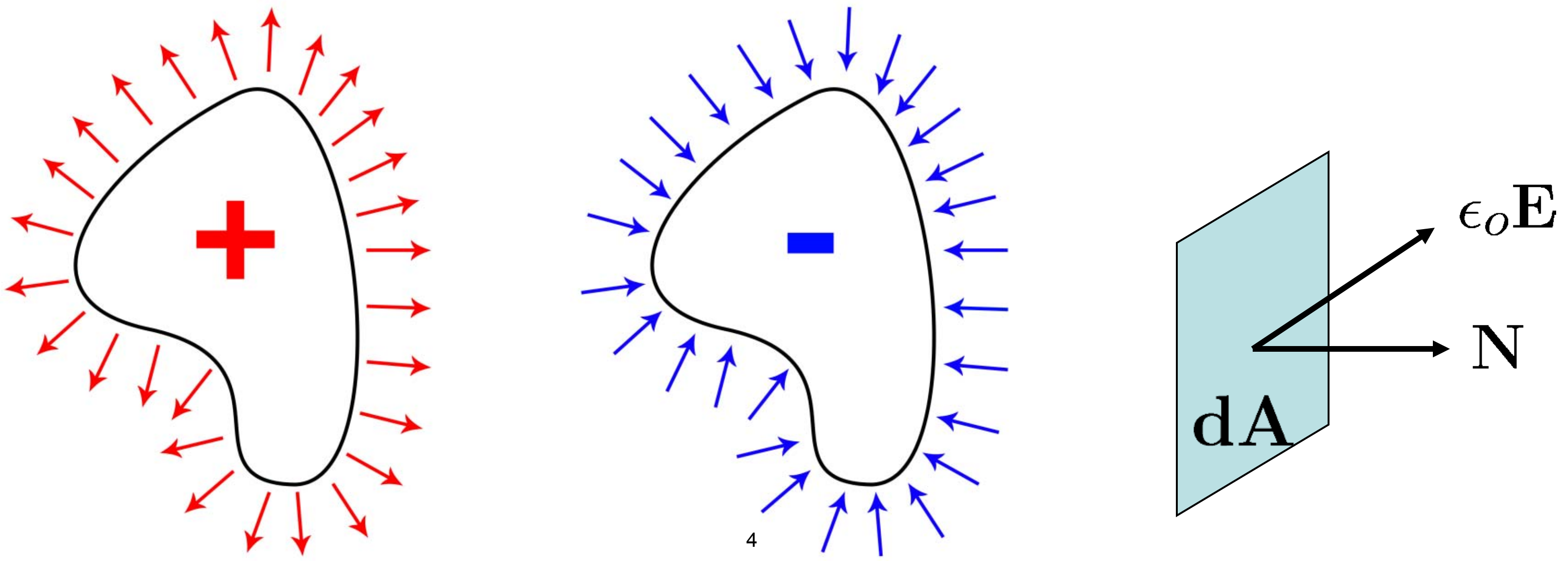
$$\vec{J} = \rho_+ \vec{v}_+ + \rho_- \vec{v}_-$$

$$\rho = \rho_+ + \rho_-$$

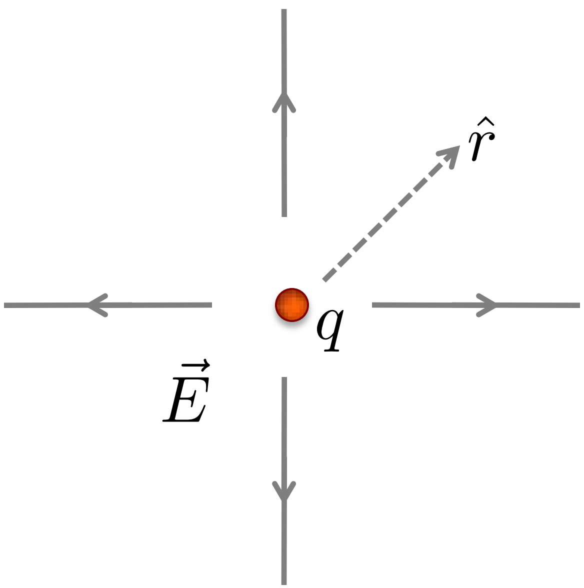
Gauss' Law

Flux of $\epsilon_o \mathbf{E}$ through closed surface ***S*** = net charge inside V

$$\int_S \epsilon_o \mathbf{E} \cdot \mathbf{dA} = \int_{\mathbf{V}} \rho \mathbf{dV} = Q_{enclosed}$$

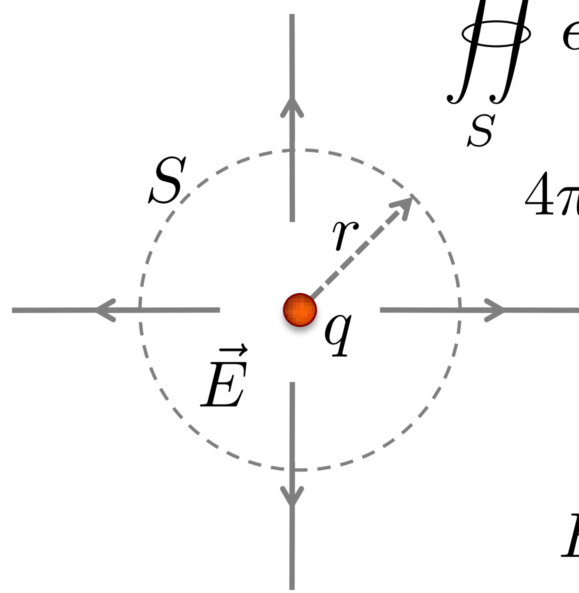


Point Charge Example



Apply Gauss' Law in integral form making use of symmetry to find $\vec{E}$

$$\oiint_S \epsilon_o \vec{E} \cdot dS = q$$



$$4\pi\epsilon_o E_r r^2 = q$$

$$E_r = \frac{q}{4\pi\epsilon_o r^2}$$

- Assume that the image charge is uniformly distributed at $\mathrm{r} = \infty$. Why is this important ?
- Symmetry

$$\Rightarrow \vec{E} = E_r(r)\,\hat{r}$$

Gauss' Law Tells Us …

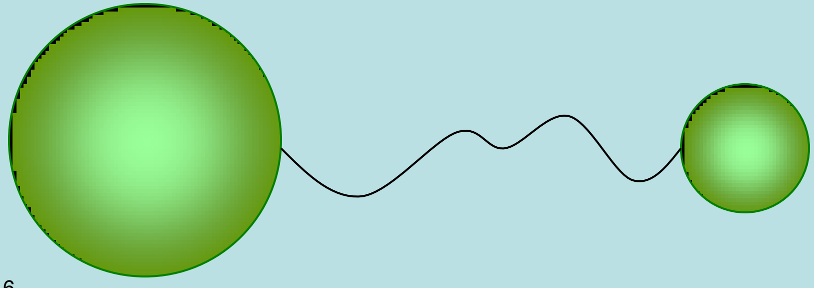
… the electric charge can reside only on the surface of the conductor.

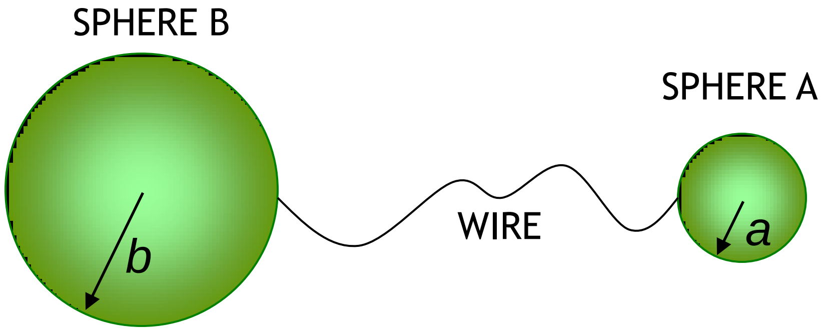
[If charge was present inside a conductor, we can draw a Gaussian surface around that charge and the electric field in vicinity of that charge would be non-zero ! A non-zero field implies current flow through the conductor, which will transport the charge to the surface.]

… there is no charge at all on the inner surface of a hollow conductor.

… that, if a charge carrying body has a sharp point, then the electric field at that point is much stronger than the electric field over the smoother part of the body.

Lets show this by considering two spheres of different size, connected by a long, thin wire …





Because the two spheres are far apart, we can assume that charges are uniformly distributed across the surfaces of the two spheres, with charge q_a *on the surface of sphere A and* q_b *on the surface of sphere B*

$$q_a + q_b = q$$

$$V_b = \frac{q_b}{4\pi\epsilon b} \qquad V_a = \frac{q_a}{4\pi\epsilon a}$$

$$since \quad V_b = V_a \quad then$$

$$q_b = q\frac{b}{a+b} \qquad q_a = q\frac{a}{a+b}$$

… and the E-field on the surface of the spheres is:

$$E_b = \frac{q_b}{4\pi\epsilon b^2} = \frac{q}{4\pi\epsilon(a+b)b} \qquad E_a = \frac{q_a}{4\pi\epsilon a^2} = \frac{q}{4\pi\epsilon(a+b)a}$$

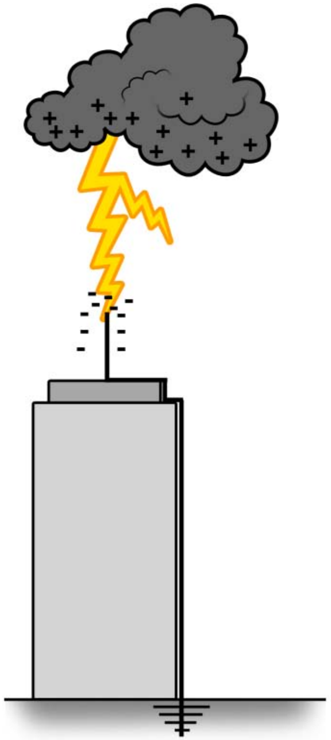
Note that $E_a >> E_b$ if $b >> a$

from Shen and Kong

Lightning Rod

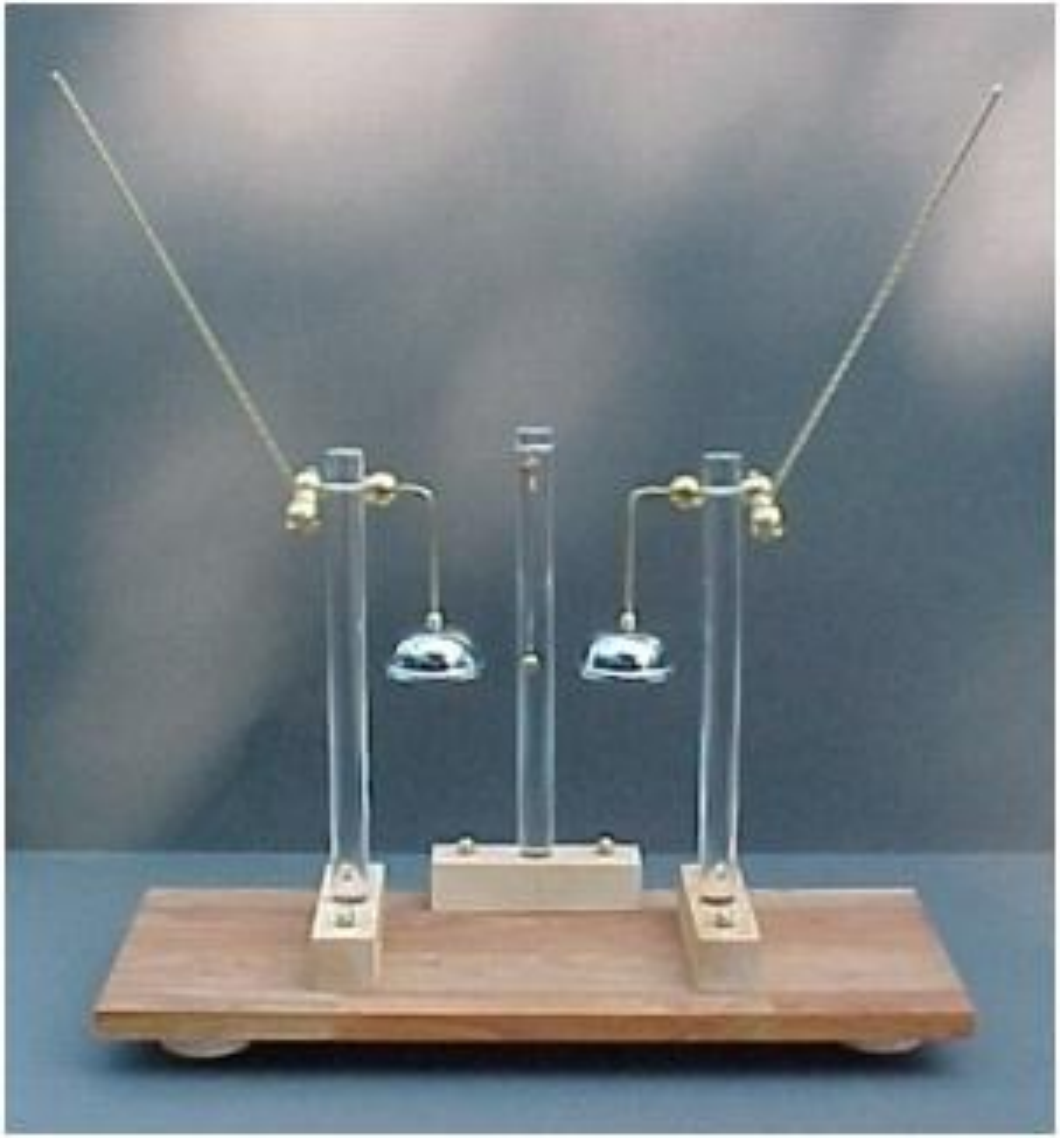
Image by http://www.flickr.com/photos/zokuga/5817408342/ on flickr

When a conductive body contains sharp points, the electric field on these points is much stronger than that on the smooth part of the conducting body.



Lighting Rods are connected to the ground. When a cloud carrying electric charges approaches, the rod attracts opposite charges from the ground. The Electric field at the tip of the rod is much stronger than anywhere else. When the E-field exceeds the air breakdown strength (of 33 kV/cm), charges start to travel to ground.

Modern demo of Gordon's bells [Andrew Gordon (1742)]

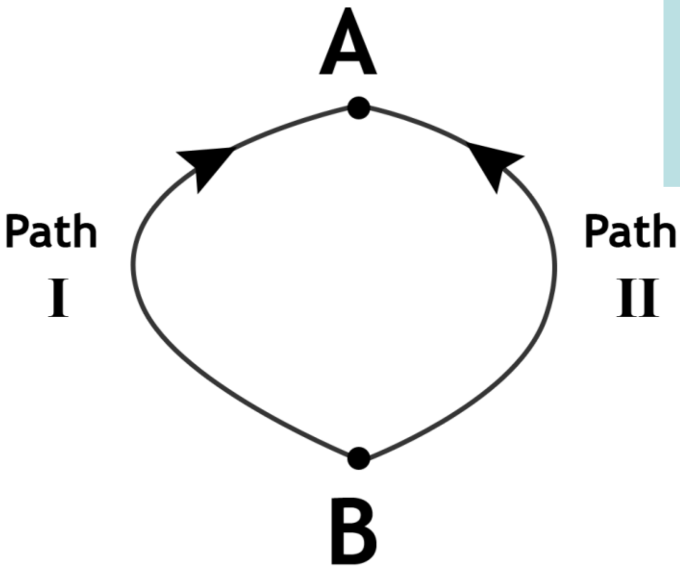


http://www.arcsandsparks.com/franklin.html

Faraday's Law

Dynamic form: $\nabla \times \vec{E} = \frac{\partial}{\partial t}\mu_o \vec{H}$

Static form: $\nabla \times \vec{E} = 0 \quad \Rightarrow \vec{E} = -\nabla\Phi$ and $\Phi_a - \Phi_b = \int_a^b \vec{E}\cdot d\vec{C}$



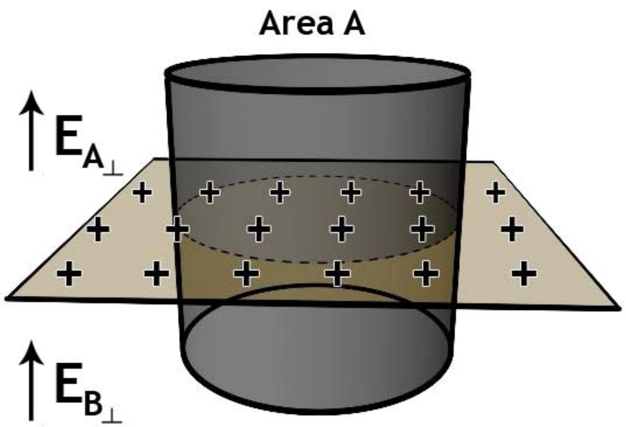
$$\oint \vec{E}\cdot d\vec{C} = 0 \quad (KVL)$$

$$\Rightarrow \underbrace{\int_b^a \vec{E}\cdot d\vec{C}}_{\text{Path I}} + \underbrace{\int_a^b \vec{E}\cdot d\vec{C}}_{\text{Path II}} = 0$$

$$\Rightarrow \underbrace{\int_b^a \vec{E}\cdot d\vec{C}}_{\text{Path I}} = \underbrace{\int_a^b \vec{E}\cdot d\vec{C}}_{\text{Path II}}$$

A unique path-independent potential may be defined if and only if $\frac{\partial B}{\partial t} = 0$

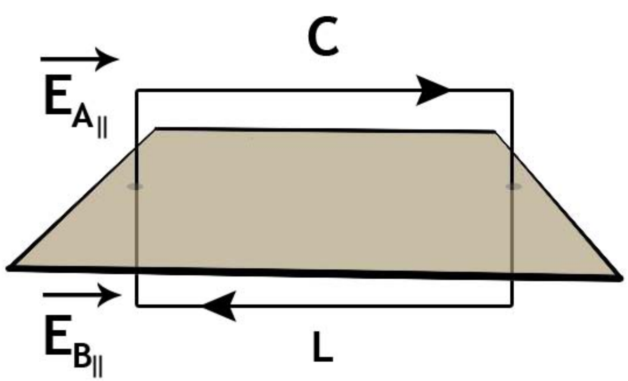
Boundary Conditions



$$\lim_{\delta \to 0} \; Gauss \Rightarrow (\epsilon_0 E_{A\perp} - \epsilon_0 E_{B\perp})A = \rho_s A$$

$$\hat{n} \cdot (\epsilon_0 E_A - \epsilon_0 E_B) = \rho_s$$

Normal $\vec{E}$ is discontinuous at a surface charge.

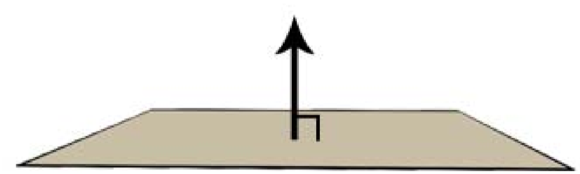


$$\lim_{\delta \to 0} \; Faraday \Rightarrow (E_{A\parallel} - E_{B\parallel})L = 0$$

$$\hat{n} \times (E_A - E_B) = 0$$

Tangential $\vec{E}$ is continuous at a surface.

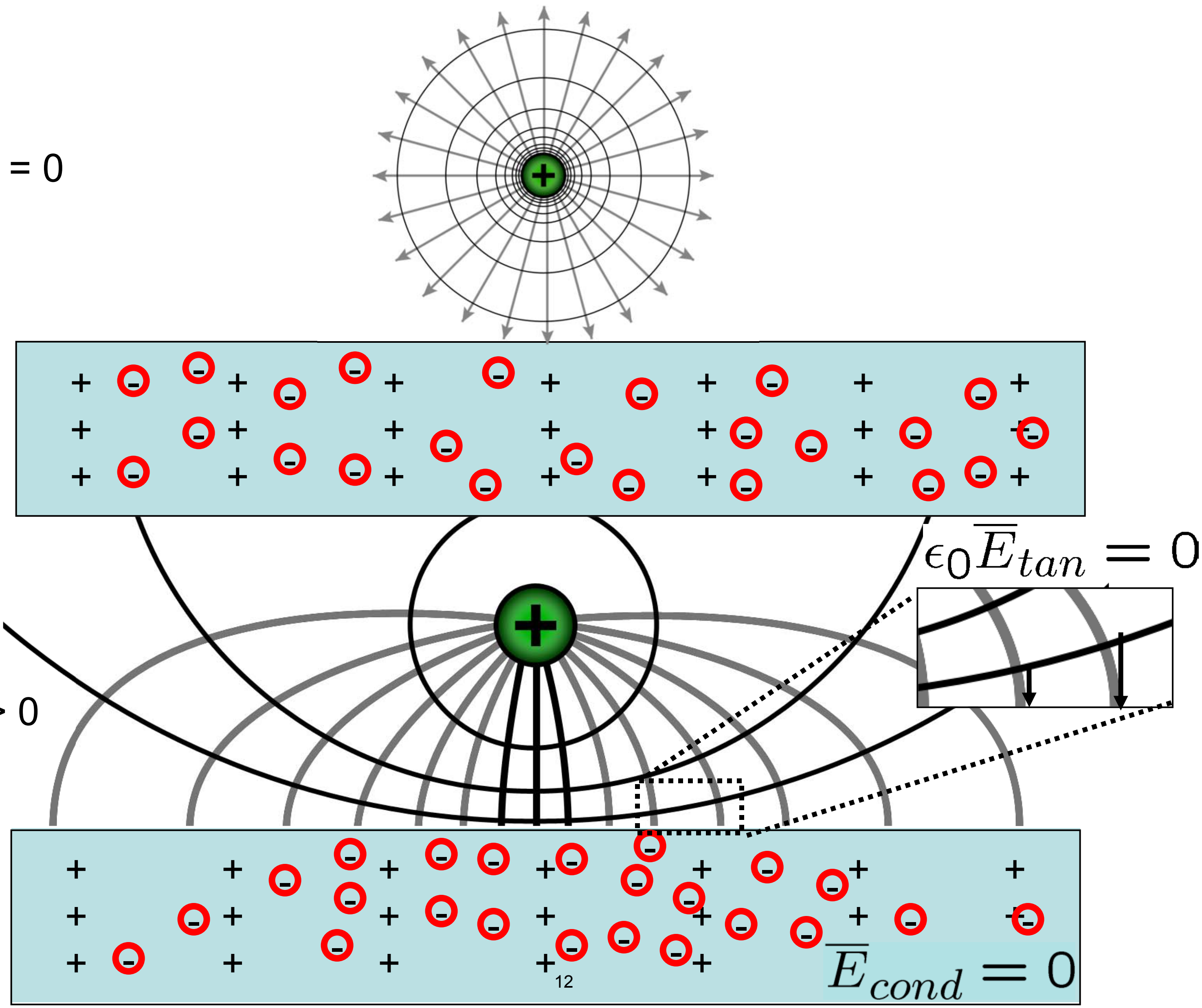
A static field terminates perpendicularly on a conductor



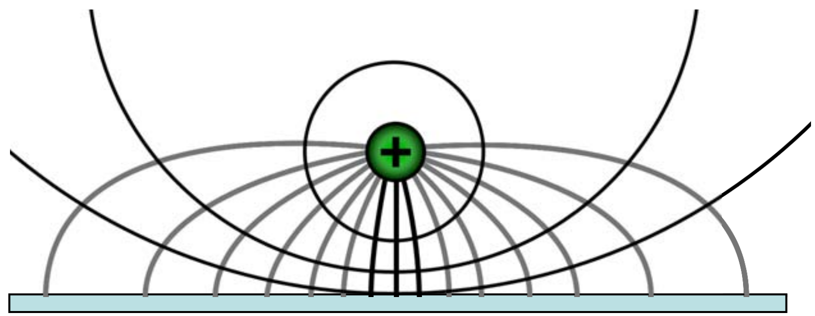
Point Charges Near Perfect Conductors

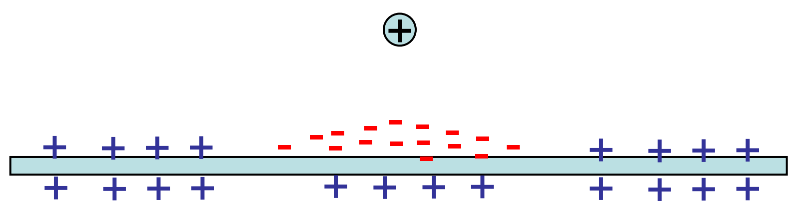
Time t = 0

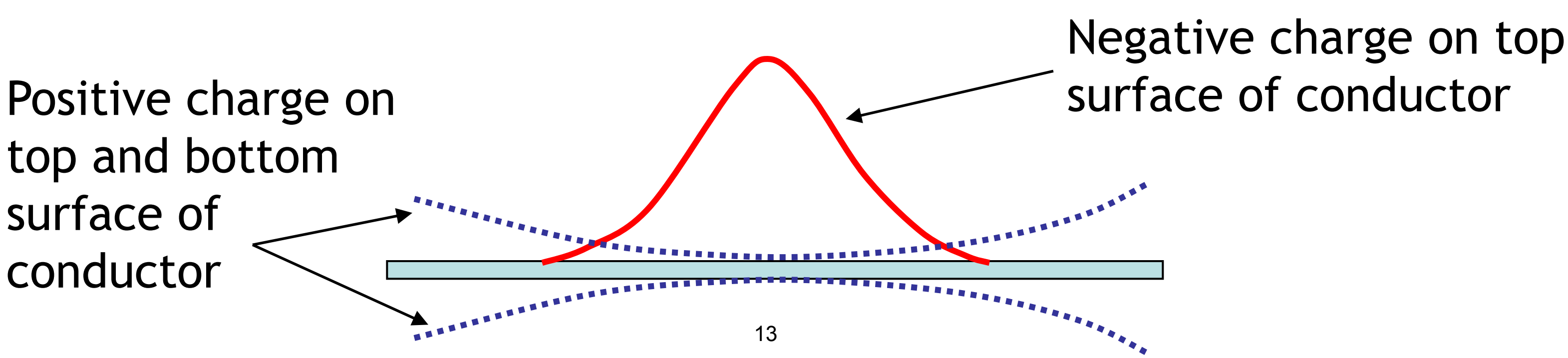
Time t >> 0



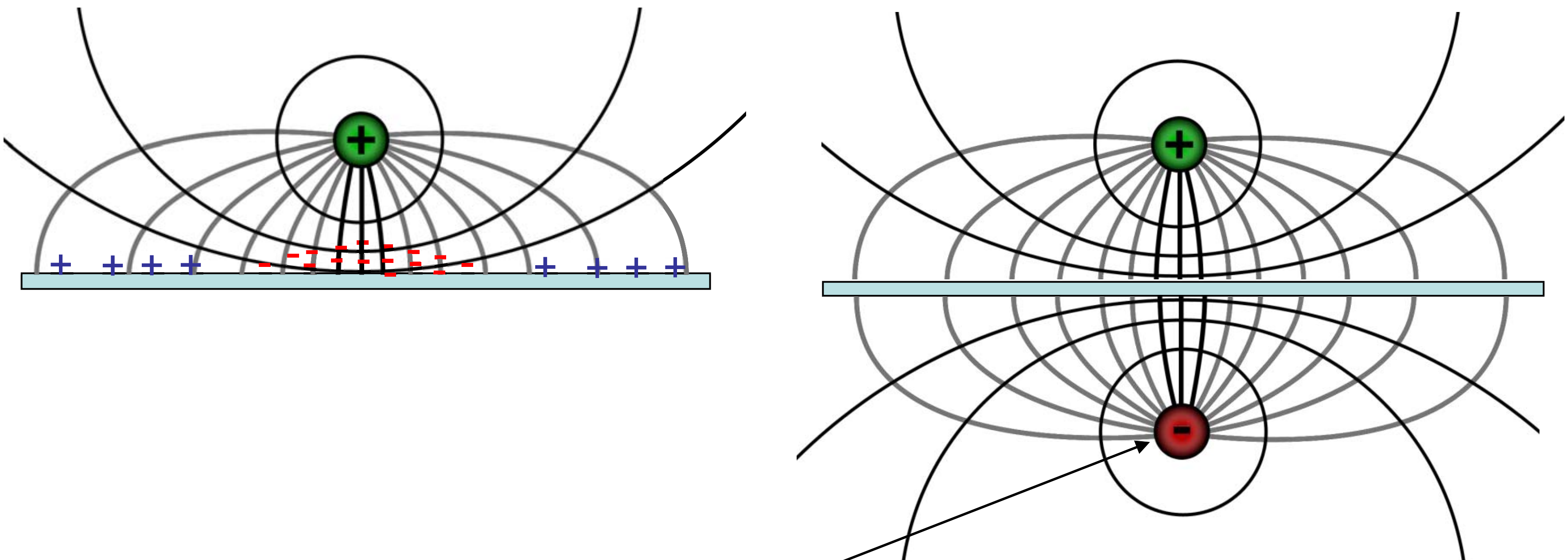
Point Charges Near Perfect Conductors







Uniqueness and Equivalent Image Charges

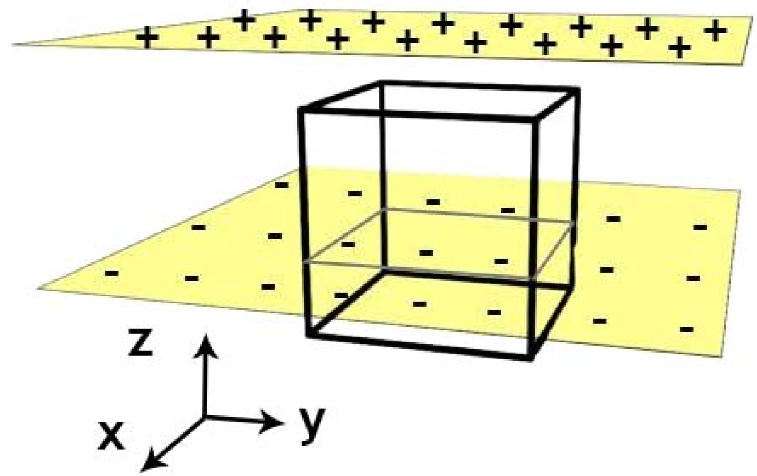


Equivalent Image Charge

Electrostatic Boundary Conditions

$$\overline{n} \cdot \left(\epsilon_0 \overline{E}^a - \epsilon_0 \overline{E}^b \right) = \sigma_s$$

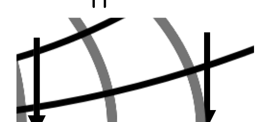
There is a jump in the normal electric field as one passes through a surface charge



$$\overline{n} \times \left(\overline{E}^a - \overline{E}^b \right) = 0$$

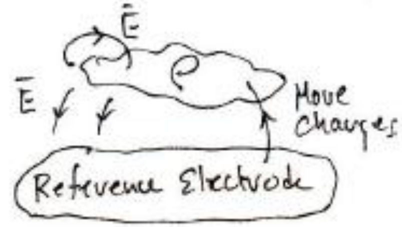
Tangential field is continuous

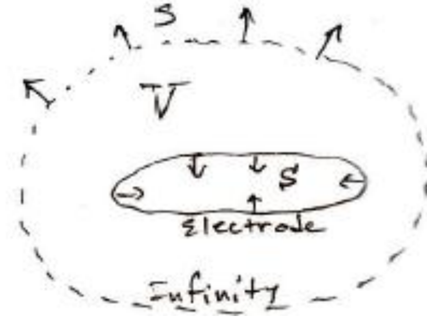
$$\epsilon_0 \overline{E}_{||} = 0$$



$$\overline{E}_{cond} = 0$$

Energy Stored in Electric Fields





1. Begin with a neutral reference conductor, the charge reservoir. Its potential is zero, by definition.

2. Move charges from the reference conductor into free space, thereby creating an electric field and doing work in the process. The work is stored as potential energy in the electric fields.

3. Account for all the work done, and thereby derive the energy stored in the electric fields.

4. The argument directly extends to systems with multiple conductors (and dielectrics).

- The work done by moving charge δq to a location with potential ϕ is $\phi\,\delta q$. More generally, the work done to make an incremental charge change to a charge density is

$$\delta w = \iiint_V \phi\,\delta\rho\,dV$$

- Gauss' Law $\Rightarrow$ $\quad \delta\rho = \nabla\cdot\delta\epsilon_o\vec{E} \quad \Rightarrow$

$$\begin{aligned}
\delta w &= \iiint_V \phi\,\nabla\cdot\delta\epsilon_o\vec{E}\,dV \\
&= \iiint_V \left[\nabla\cdot(\phi\,\delta\epsilon_o\vec{E}) - \delta\epsilon_o\vec{E}\cdot\nabla\phi\right]dV \\
&= \underbrace{\oiint_S \phi\,\delta\epsilon_o\vec{E}\cdot d\vec{S}}_{\text{ZERO ! WHY ?}} + \iiint_V \vec{E}\cdot\delta\epsilon_o\vec{E}\,dV
\end{aligned}$$

ENERGY DENSITY [J/m³]

$$\frac{\delta W}{\delta V} = \vec{E}\cdot\epsilon_o\delta\vec{E} \quad \Rightarrow \quad \frac{W}{Volume} = \frac{1}{2}\epsilon_o E^2$$

Energy Stored in Electric Fields

The energy stored in an electric field is $\frac{1}{2}\,\varepsilon_0 E^2$.
The maximum achievable field strength is
typically limited by electric breakdown

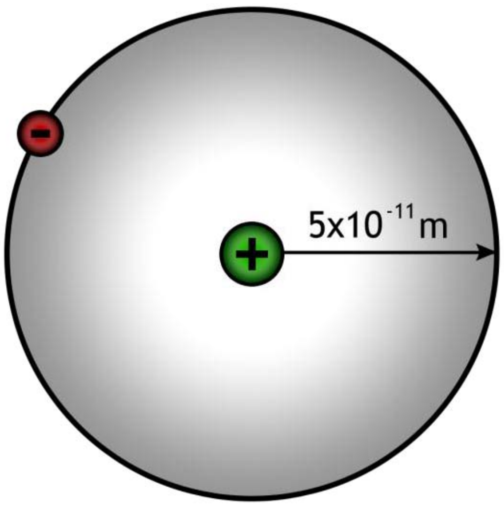
- Large air gaps: $E \leq 10^6$ V/m (approximately) $\rightarrow$ $W = 4.4$ J/m^3
- Micron-sized air gaps: $E \leq 10^8$ V/m (approximately) $\rightarrow$ $W = 44$ kJ/m^3
- Biological gaps: $E \leq 10^9$ V/m (approximately) $\rightarrow$ $W = 4.4$ MJ/m^3
- Gasoline: 38 GJ/m^3

Note: Dielectric constant is

$$\epsilon_o = 8.854 \times 10^{-12}\, F/m$$

Modeling Atoms and Molecules as Capacitors that Store Energy

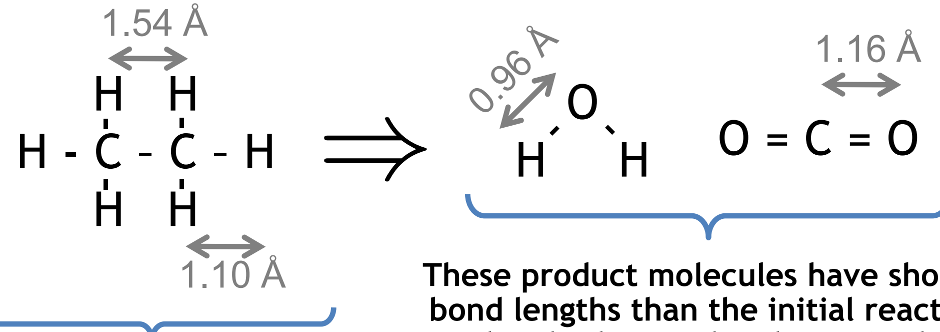
$$\phi(r) = \frac{q}{4\pi\epsilon_o r}$$



Hydrogen Atom

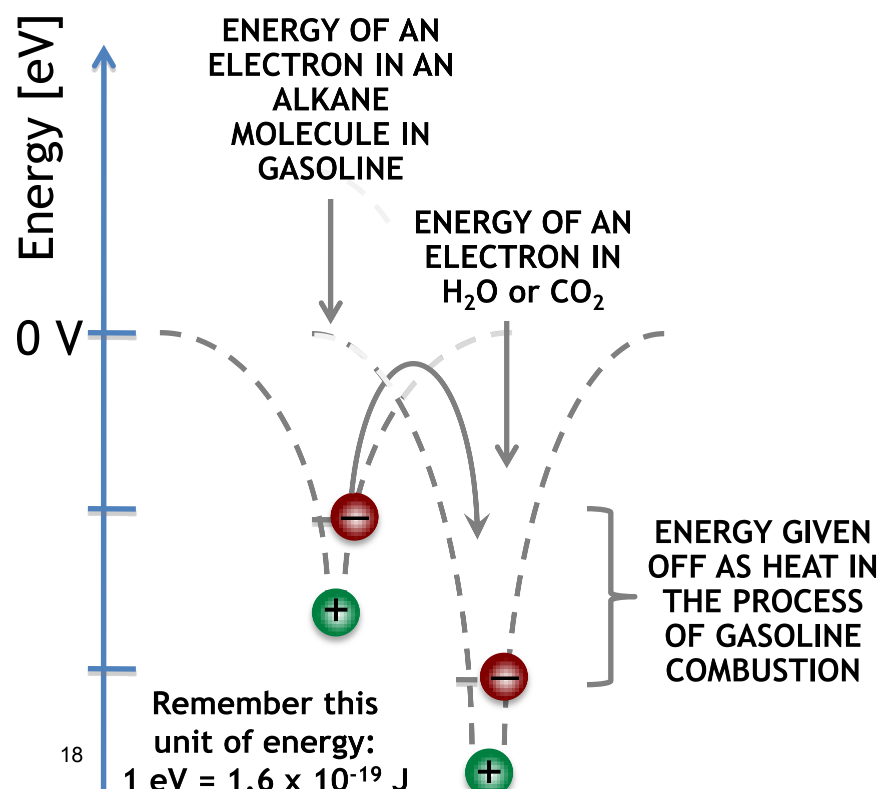
Hydrogen ground state energy is -13.6 eV

If the hydrogen radius was twice as long, what would be the ground state energy ?



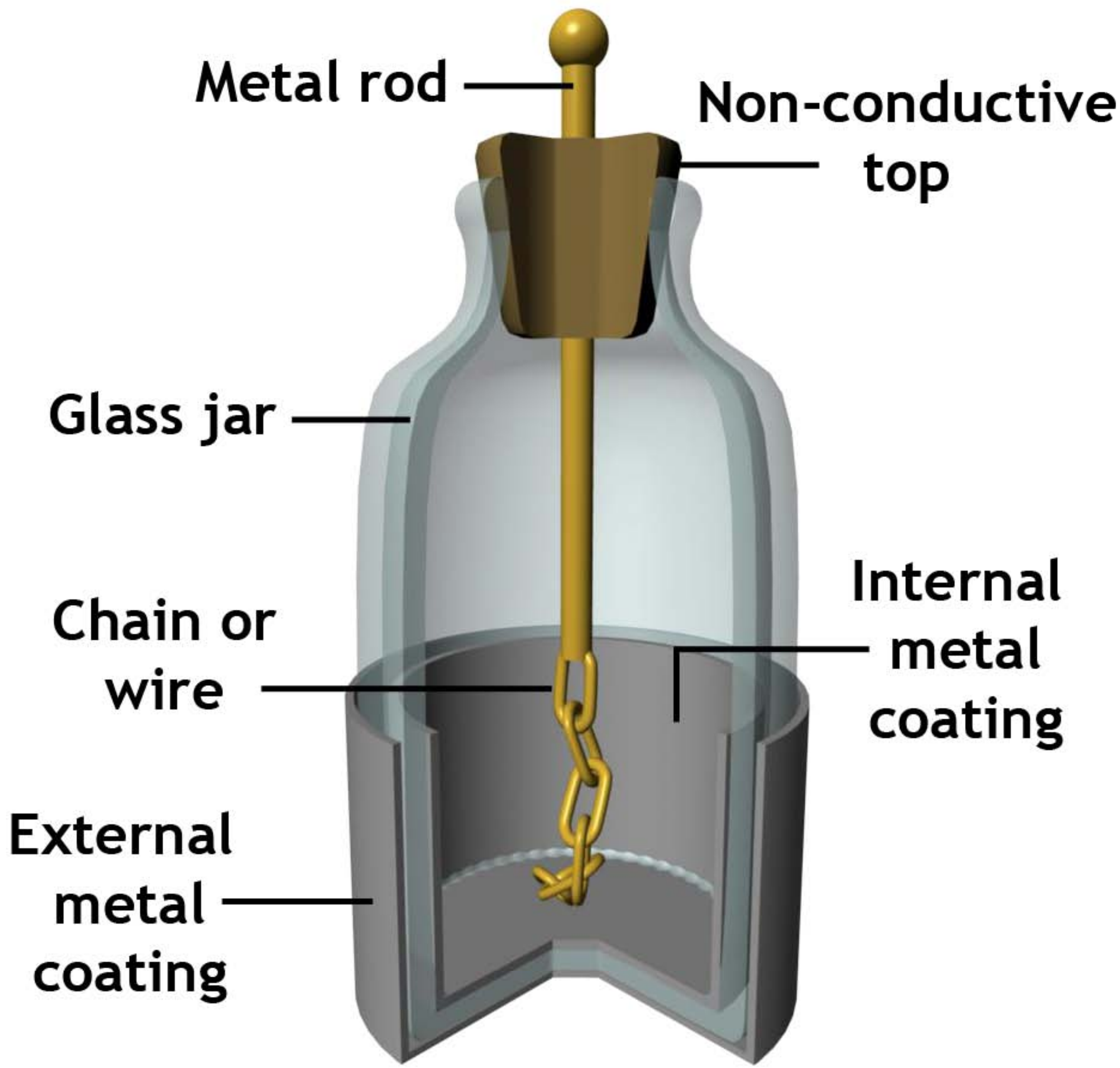
Burn this molecule by reacting it with oxygen

These product molecules have shorter bond lengths than the initial reactant molecule, hence the charge in them sits closer together. This can be modeled as a higher capacitance. Since voltage V = Q/C = E • d is reduced, stored energy W/Volume = ½ $\varepsilon_o E^2$ is reduced in these molecules

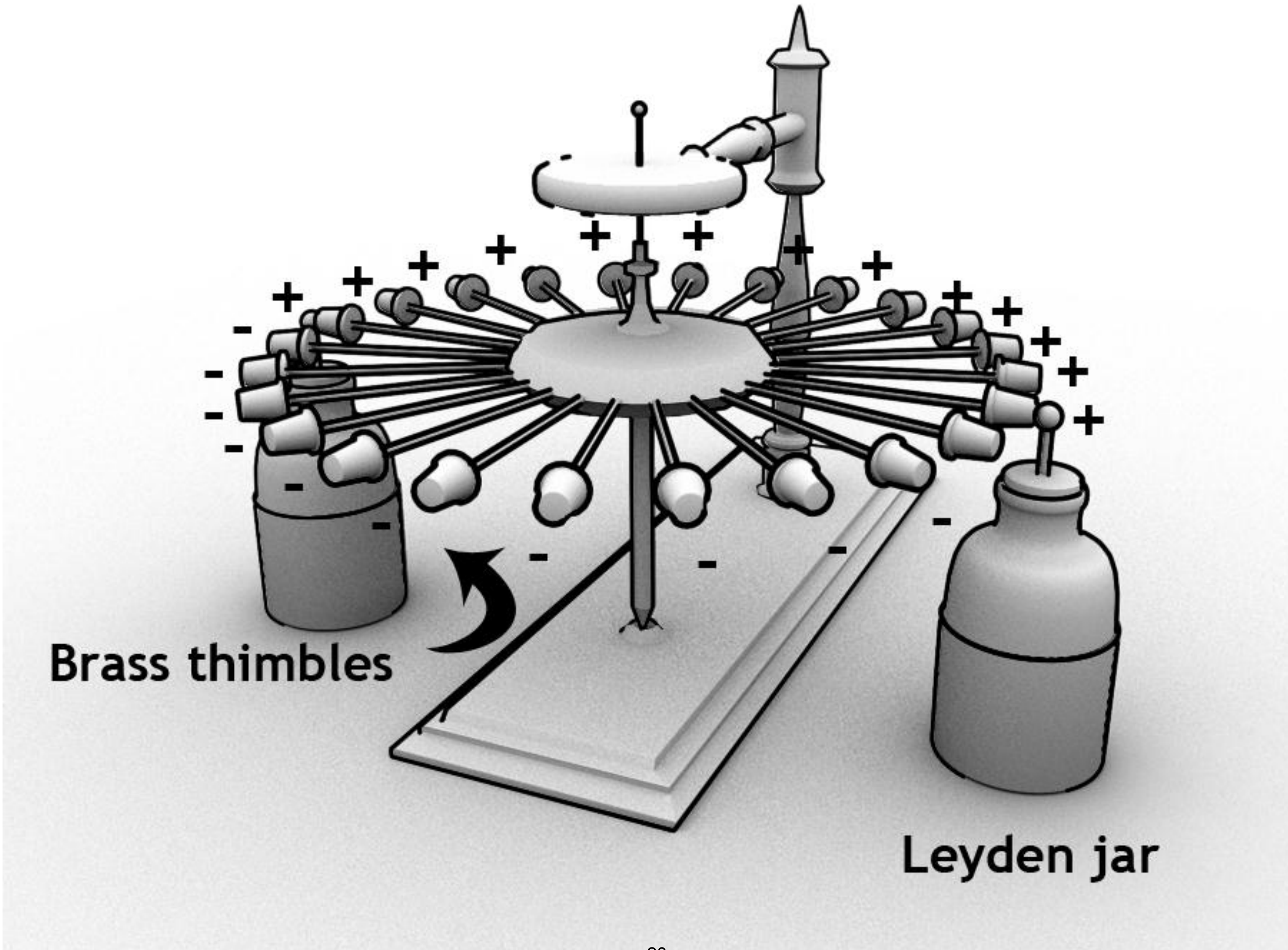


Remember this unit of energy: 1 eV = 1.6 x 10^{-19} J

Leyden Jar


Metal rod
Non-conductive top
Glass jar
Chain or wire
Internal metal coating
External metal coating

Franklin's motor (1748)



KEY TAKEAWAYS

- Maxwell's Equations (in Free Space with Electric Charges present):

	DIFFERENTIAL FORM	INTEGRAL FORM
E-Gauss:	$\nabla \cdot \epsilon_o \vec{E} = \rho$	$\oiint_S \epsilon_o \vec{E} \cdot dS = \iiint_V \rho dV$
Faraday:	$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\mu_o \vec{H}$	$\oint_C \vec{E} \cdot d\vec{C} = \iint_S -\frac{\partial}{\partial t}\mu_o \vec{H} \cdot d\vec{S}$
H-Gauss:	$\nabla \cdot \mu_o \vec{H} = 0$	$\oiint_S \mu_o \vec{H} \cdot d\vec{S} = 0$
Ampere:	$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\epsilon_o \vec{E}$	$\oint_C \vec{H} \cdot d\vec{C} = \iint_S (\vec{J} + \frac{\partial}{\partial t}\mu_o \vec{H}) \cdot d\vec{S}$

- Boundary conditions for E-field:
  . Normal E-field – discontinuous
  . Tangential E-Field - continuous

GOOD FACTS TO REMEMBER:

AIR BREAKDOWN STRENGTH is 33 kV/cm
NEW UNIT OF ENERGY: $1\, eV = 1.6 \times 10^{-19} J$

- Energy stored in the electric field per unit volume is: $\frac{W}{Volume} = \frac{1}{2}\epsilon_o E^2$

- Energy released when fuel molecules are oxidized since the charges in the products are positioned closer together than in reactants (hence in a lower energy state)

- Dielectric constant in free space is $\epsilon_o = 8.854 \times 10^{-12}\ F/m$

6.007 Electromagnetic Energy: From Motors to Lasers
Spring 2011