Lecture 8: Expected Utility Theory

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The Plan

Course so far introduced basic theory of choice and utility, extended to consumer and producer theory.

Last topic extends in another direction: choice under uncertainty

Choice under Uncertainty

All choices made under some kind of uncertainty.

Sometimes useful to ignore uncertainty, focus on ultimate choices. Other times, must model uncertainty explicitly.

Examples:

- Insurance markets.
- Financial markets.
- Game theory.

Overview

Impose extra assumptions on basic choice model of Lectures 1–2.

Rather than choosing outcome directly, decision-maker chooses **uncertain prospect** (or **lottery**).

A lottery is a probability distribution over outcomes.

Leads to **von Neumann-Morgenstern expected utility model**.

Next three lectures: applications/extensions.

1. Measures of risk-aversion.
2. Comparison of uncertain prospects.
3. Critiques/extensions of expected utility theory.

Consequences and Lotteries

Two basic elements of expected utility theory: **consequences** (or **outcomes**) and **lotteries.**

Consequences

Finite set C of consequences.

Consequences are what the decision-maker ultimately cares about.

Example: "I get pneumonia, my health insurance company covers most of the costs, but I have to pay a $500 deductible."

Consumer does not choose consequences directly.

Lotteries

Consumer chooses a lottery, p.

Lotteries are probability distributions over consequences:
$p : C \to [0, 1]$ with $\sum_{c \in C} p(c) = 1$.

Set of all lotteries is denoted by P.

Example: "A gold-level health insurance plan, which covers all kinds of diseases, but has a $500 deductible."

Makes sense because consumer assumed to rank health insurance plans only insofar as lead to different probability distributions over consequences.

Choice

In Lectures 1–2, decision-maker make choices from set of alternatives X.

What's set of alternatives here, C or P?

Answer: P
Consumer does not choose consequences directly, but instead chooses lotteries.

Just like in Lectures 1–2, assume decision-maker has a rational preference relation $\succsim$ on P.

Convex Combinations of Lotteries

Given two lotteries p and p', the convex combination $\alpha p + (1-\alpha)\,p'$ is the lottery defined by

$$\left(\alpha p + (1-\alpha)\,p'\right)(c) = \alpha p\,(c) + (1-\alpha)\,p'\,(c) \text{ for all } c \in C.$$

One way to generate it:

- **First**, randomize between p and p' with weights α and $1-\alpha$.
- **Second**, choose a consequence according to whichever lottery came up.

Such a probability distribution over lotteries is called a **compound lottery**.

In expected utility theory, **no distinction** between simple and compound lotteries: simple lottery $\alpha p + (1-\alpha)\,p'$ and above compound lottery give same distribution over consequences, so identified with same element of P.

The Set P

As $\alpha p + (1 - \alpha) p'$ is a lottery, P is convex.

P is also closed and bounded.

$\implies P$ is a compact subset of $\mathbb{R}^n$, where $n = |C|$.

Utility

Just like in Lectures 1–2, whenever $\succsim$ is rational and continuous, can be represented by continuous utility function $U : P \to \mathbb{R}$:

$$p \succsim q \iff U(p) \geq U(q)$$

Intuitively, want more than this.

Want not only that consumer has utility function over **lotteries**, but also that somehow related to "utility" over **consequences**.

Only care about lotteries insofar as affect distribution over consequences, so preferences over lotteries should have something to do with "preferences" over consequences.

Expected Utility

Best we could hope for is representation by utility function of following form:

Definition

A utility function $U : P \to \mathbb{R}$ has an **expected utility form** if there exists a function $u : C \to \mathbb{R}$ such that

$$U(p) = \sum_{c \in C} p(c)\, u(c) \text{ for all } p \in P.$$

In this case, the function U is called an **expected utility function**, and the function u is call a **von Neumann-Morgenstern utility function**.

If preferences over lotteries happen to have an expected utility representation, it's **as if** consumer has a "utility function" over consequences (and chooses among lotteries so as to maximize expected "utility over consequences").

Expected Utility: Remarks

$$U(p) = \sum_{c \in C} p(c)\, u(c)$$

Expected utility function $U : P \to \mathbb{R}$ represents preferences $\succsim$ on P just like in Lectures 1–2.
$U : P \to \mathbb{R}$ is an example of a standard utility function.

von Neumann-Morgenstern utility function $u : C \to \mathbb{R}$ is **not** a standard utility function.

Can't have a "real" utility function on consequences, as consumer never chooses among consequences.

If preferences over lotteries happen to have an expected utility representation, it's **as if** consumer has a "utility function" over consequences.

This "utility function" over consequences is the von Neumann-Morgenstern utility function.

Example

Suppose hipster restaurant doesn't let you order steak or chicken, but only probability distributions over steak and chicken.

How should you assess menu item $(p\,(steak)\,, p\,(chicken))$?

One way: ask yourself how much you'd like to eat steak, $u\,(steak)$, and chicken, $u\,(chicken)$, and evaluate according to

$$p\,(steak)\,u\,(steak) + p\,(chicken)\,u\,(chicken)$$

If this is what you'd do, then your preferences have an expected utility representation.

Example (continued)

Suppose instead you choose whichever menu item has $p\,(steak)$ closest to $\frac{1}{2}$.

Your preferences are rational, so they have a utility representation.

But they do not have an expected utility representation.

Property of EU: Linearity in Probabilities

If $U : P \to \mathbb{R}$ is an expected utility function, then

$$U\left(\alpha p + (1-\alpha)\, p'\right) = \alpha U(p) + (1-\alpha)\, U\left(p'\right)$$

In fact, a utility function $U : P \to \mathbb{R}$ has an expected utility form iff this equation holds for all p, p', and $\alpha \in [0, 1]$.

Exercise: prove it. (See MWG for help.)

Property of EU: Invariant to Affine Transformations

Suppose $U : P \to \mathbb{R}$ is an expected utility function representing preferences $\succsim$.

Any increasing transformation of U also represents $\succsim$.

Not all increasing transformations of U have expected utility form.

Theorem

Suppose $U : P \to \mathbb{R}$ is an expected utility function representing preferences $\succsim$. Then $V : P \to \mathbb{R}$ is also an expected utility function representing $\succsim$ iff there exist $a, b > 0$ such that

$$V(p) = a + bU(p) \text{ for all } p \in P.$$

If this is so, we also have $V(p) = \sum_{c \in C} p(c)\, v(c)$ for all $p \in P$, where

$$v(c) = a + bu(c) \text{ for all } c \in C.$$

What Preferences have an Expected Utility Representation?

Preferences must be rational to have any kind of utility representation.

Preferences on a compact and convex set must be continuous to have a continuous utility representation.

Besides rationality and continuity, what's needed to ensure that preferences have an expected utility representation?

The Independence Axiom

Definition

A preference relation $\succsim$ satisfies **independence** if, for every $p, q, r \in P$ and $\alpha \in (0, 1)$,

$$p \succsim q \iff \alpha p + (1 - \alpha) r \succsim \alpha q + (1 - \alpha) r.$$

Can interpret as form of "dynamic consistency."

Back to Example

Suppose choose lottery with $p\left(steak\right)$ closest to $\frac{1}{2}$.

Let $p = \left(\frac{1}{2}, \frac{1}{2}\right)$, $q = (0, 1)$, $r = (1, 0)$, and $\alpha = \frac{1}{2}$.

Then

$$p = \left(\frac{1}{2}, \frac{1}{2}\right) \succ (0, 1) = q$$

but

$$\alpha q + (1 - \alpha)\, r = \left(\frac{1}{2}, \frac{1}{2}\right) \succ \left(\frac{3}{4}, \frac{1}{4}\right) = \alpha p + (1 - \alpha)\, r$$

Does not satisfy independence.

Expected Utility: Characterization

Theorem (Expected Utility Theorem)

A preference relation $\succsim$ has an expected utility representation iff it satisfies rationality, continuity, and independence.

Intuition: both having expected utility form and satisfying independence boil down to having straight, parallel indifference curves.

Subjective Expected Utility Theory

So far, probabilities are objective.

In reality, uncertainty is usually subjective.

Subjective expected utility theory (Savage, 1954): under assumptions roughly similar to ones form this lecture, preferences have an expected utility representation where both the utilities over consequences **and the subjective probabilities themselves** are revealed by decision-maker's choices.

Thus, expected utility theory applies even when the probabilities are not objectively given.

To learn more, a good starting point is Kreps (1988), "Notes on the Theory of Choice."