Lectures 1–2: Choice, Preference, and Utility

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14.121

Individual Decision-Making

Economics studies interaction of individual decision-makers.

14.121: theory of individual choice

Rest of micro sequence: how individuals interact in markets and other settings

Utility Maximization

Basic model of individual choice:

- A decision-maker (DM) must choose one **alternative** x from a set X.
- Chooses to maximize a **utility function** u.
- u specifies how much utility DM gets from each alternative:

$$u : X \longrightarrow \mathbb{R}$$

Example: DM chooses whether to eat an apple or a banana.

$X = \{apple, banana\}$.

Utility function might say $u\,(apple) = 7$, $u\,(banana) = 12$.

What do Utility Levels Mean? Hedonic Interpretation

Utility is an objective measure of individual's well-being.

> *Nature has placed mankind under the governance of two sovereign masters, pain and pleasure. It is for them alone to point out what we ought to do... By the principle of utility is meant that principle which approves or disapproves of every action whatsoever according to the tendency it appears to have to augment or diminish the happiness of the party whose interest is in question: or, what is the same thing in other words to promote or to oppose that happiness. I say of every action whatsoever, and therefore not only of every action of a private individual, but of every measure of government.*

" $u\,(apple) = 7$, $u\,(banana) = 12$" $=$ apple gives 7 units of pleasure, banana gives 12 units of pleasure.

This is **not** the standard way economists think about utility.

What do Utility Levels Mean? Revealed-Preference Interpretation

Utility represents an individual's choices.

- Individual choices are **primitive data** that economists can observe.
- Choices are taken to **reveal** individual's preferences.
- Utility is a **convenient mathematical construction** for modeling choices and preferences.

"$u\left(apple\right)=7,\ u\left(banana\right)=12$" $=$ individual prefers bananas to apples.
"$u\left(apple\right)=2,\ u\left(banana\right)=15$" $=$ individual prefers bananas to apples.

Today's lecture: how does this work?

Choice

How can an individual's choices reveal her preferences?

A **choice structure** (or **choice dataset**) $(\mathcal{B}, C)$ consists of:

1. A set $\mathcal{B}$ of **choice sets** $B \subseteq X$.
2. A **choice rule** C that maps each $B \in \mathcal{B}$ to a non-empty set of chosen alternatives $C(B) \subseteq B$.

Interpretation: $C(B)$ is the set of alternatives the DM might choose from B.

Preference

Goal: relate observable choice data to preferences over X.

A **preference relation** $\succsim$ is a binary relation on X.

"$x \succsim y$" means "x is weakly preferred to y.'

Given preference relation $\succsim$, define:

- **Strict preference** ($\succ$): $x \succ y \Leftrightarrow x \succsim y$ but not $y \succsim x$.
- **Indifference ($\sim$)**: $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$.

Rational Preferences

To make any progress, need to impose some restrictions on preferences.

Most important: **rationality**

Definition

A preference relation $\succsim$ is **rational** if it satisfies:

1. **Completeness:** for all x, y, $x \succsim y$ or $y \succsim x$.
2. **Transitivity:** for all x, y, z, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

If $\succsim$ is rational, then $\succ$ and $\sim$ are also transitive.

Hard to say much about behavior of irrational DM.

Maximizing a Preference Relation

Optimal choices according to $\succsim$:

$$C^* (B, \succsim) = \{x \in B : x \succsim y \text{ for all } y \in B\}$$

$\succsim$ **rationalizes** choice data $(\mathcal{B}, C)$ if $C(B) = C^*(B, \succsim)$ for all $B \in \mathcal{B}$

Fundamental Question of Revealed Preference Theory

When does choice data reveal that individual is choosing according to rational preferences?

Definition

Given choice data $(\mathcal{B}, C)$, the **revealed preference relation** $\succsim^*$ is defined by

$$x \succsim^* y \Leftrightarrow \text{there is some } B \in \mathcal{B} \text{ with } x, y \in B \text{ and } x \in C(B)$$

x is **weakly revealed preferred** to y if x is ever chosen when y is available.

x is **strictly revealed preferred** to y if there is some $B \in \mathcal{B}$ with $x, y \in B$, $x \in C(B)$, and $y \notin C(B)$.

WARP

Key condition on choice data for $\succsim^*$ to be rational and generate observed data: **weak axiom of revealed preference (WARP)**.

Definition

Choice data $(\mathcal{B}, C)$ satisfies **WARP** if whenever there exists $B \in \mathcal{B}$ with $x, y \in B$ and $x \in C(B)$, then for all $B' \in \mathcal{B}$ with $x, y \in B'$, it is not the case that both $y \in C(B')$ and $x \notin C(B')$.

"If x is weakly revealed preferred to y, then y cannot be strictly revealed preferred to x."

WARP: Example

$X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$

Choice rule C_1: $C_1(\{x, y\}) = \{x\}$, $C_1(\{x, y, z\}) = \{x\}$.

Satisfies WARP: x is weakly revealed preferred to y and z, nothing is strictly revealed preferred to x.

Choice rule C_2: $C_2(\{x, y\}) = \{x\}$, $C_2(\{x, y, z\}) = \{x, y\}$.

Violates WARP: y is weakly revealed preferred to x, x is strictly revealed preferred to y.

A Fundamental Theorem of Revealed Preference

Theorem

If choice data $(\mathcal{B}, C)$ satisfies WARP and includes all subsets of X of up to 3 elements, then $\succsim^*$ is rational and rationalizes the data: that is, $C^*(B, \succsim^*) = C(B)$. Furthermore, this is the only preference relation that rationalizes the data.
Conversely, if the choice data violates WARP, then it cannot be rationalized by any rational preference relation.

Theorem tells us how individual's choices reveal her preferences: as long as choices satisfy WARP, can interpret choices as resulting from maximizing a rational preference relation.

Preference and Utility

Now that know how to infer preferences from choice, next step is representing preferences with a utility function.

Definition

A utility function $u : X \to \mathbb{R}$ **represents** preference relation $\succsim$ if, for all x, y,

$$x \succsim y \Leftrightarrow u(x) \geq u(y)$$

banana $\succsim$ *apple* is represented by **both**
$u(apple) = 7, u(banana) = 12$ and
$u(apple) = 2, u(banana) = 15$.

If u represents $\succsim$, so does any **strictly increasing transformation** of u.

Representing a given preference relation is an **ordinal property**. The numerical values of utility are **cardinal properties**.

What Preferences have a Utility Representation?

Theorem

Only rational preferences relations can be represented by a utility function.

Conversely, if X is finite, any rational preference relation can be represented by a utility function.

What Goes Wrong with Infinitely Many Alternatives?

Lexicographic preferences:

$X = [0, 1] \times [0, 1]$

$(x_1, x_2) \succsim (y_1, y_2)$ if either

- $x_1 > y_1$ or
- $x_1 = y_1$ and $x_2 \geq y_2$

Maximize first component. In case of tie, maximize second component.

Theorem

Lexicographic preferences cannot be represented by a utility function.

Continuous Preferences

What if rule out discontinuous preferences?

Definition

For $X \subseteq \mathbb{R}^n$, preference relation $\succsim$ is **continuous** if whenever $x^m \to x$, $y^m \to y$, and $x^m \succsim y^m$ for all m, we have $x \succsim y$.

Theorem

For $X \subseteq \mathbb{R}^n$, any continuous, rational preference relation can be represented by a utility function.

Review of Revealed Preference Theory

- If choice data satisfies WARP, can interpret as resulting from maximizing a rational preference relation.
- If set of alternatives is finite or preferences are continuous, can represent these preferences with a utility function.
- Utility function is just a convenient mathematical representation of individual's ordinal preferences.
- Utility may or may not be correlated with pleasure/avoidance of pain.

Properties of Preferences and Utility Functions

Doing useful analysis entails making assumptions.

Try to do this carefully: make clearest, simplest, least restrictive assumptions.

Understand what assumptions about utility correspond to in terms of preferences, since utility is just a way of representing preferences.

We now cover some of the most important assumptions on preferences. (And, implicitly, on choices.)

Setting/Notation

For rest of lecture, assume $X \subseteq \mathbb{R}^n$.

Example: Consumer Problem: given fixed budget, choose how much of n goods to consume

Notation: for vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,

- $x \geq y$ means $x_k \geq y_k$ for all $k = 1, \ldots, n$
- $x > y$ means $x_k \geq y_k$ for all k and $x_k > y_k$ for some k
- $x \gg y$ means $x_k > y_k$ for all k
- For $\alpha \in [0, 1]$,

$$\alpha x + (1 - \alpha) y = (\alpha x_1 + (1 - \alpha) y_1, \ldots, \alpha x_n + (1 - \alpha) y_n)$$

Monotonicity: Preferences

"All goods are desirable"

Definition

$\succsim$ is **monotone** if $x \geq y$ implies $x \succsim y$.
$\succsim$ is **strictly monotone** if $x > y$ implies $x \succ y$.

Monotonicity: Utility

If preferences are monotone, what does that mean for the utility function?

Theorem

Suppose utility function u represents preferences $\succsim$. Then:

$$\begin{aligned} u \text{ is non-decreasing} \quad &\Leftrightarrow \quad \succsim \text{ is monotone} \\ u \text{ is strictly increasing} \quad &\Leftrightarrow \quad \succsim \text{ is strictly monotone} \end{aligned}$$

Local-Nonsatiation

"No bliss points." (Not even local ones.)

Let $B_\varepsilon(x) = \{y : \|x - y\| < \varepsilon\}$.

Definition
$\succsim$ is **locally non-satiated** if for any x and $\varepsilon > 0$, there exists $y \in B_\varepsilon(x)$ with $y \succ x$.

If u represents $\succsim$, then $\succsim$ is locally non-satiated iff u has no local maximum.

Convexity

"Diversity is good."

Definition

$\succsim$ is **convex** if $x \succsim y$ and $x' \succsim y$ imply

$$\alpha x + (1-\alpha)\, x' \succsim y \text{ for all } \alpha \in (0,1)$$

$\succsim$ is **strictly convex** if $x \succsim y$, $x' \succsim y$, and $x \neq x'$ imply

$$\alpha x + (1-\alpha)\, x' \succ y \text{ for all } \alpha \in (0,1)$$

Does this make sense?
Is $\frac{1}{2}\mathit{beer} + \frac{1}{2}\mathit{wine}$ a good thing?

 We now discuss several properties of convex preferences.

Contour Sets

For $x \in X$, the **upper contour set of** x is

$$U(x) = \{y \in X : y \succsim x\}.$$

Theorem

$\succsim$ is convex iff $U(x)$ is a convex set for every $x \in X$.

That's why convex preferences are called convex: for every x, the set of all alternatives preferred to x is convex.

Set of Maximizers

Theorem

If $\succsim$ is convex, then for any convex choice set B, the set $C^*(B, \succsim)$ is convex.
If $\succsim$ is strictly convex, then for any convex choice set B, the set $C^*(B, \succsim)$ is single-valued (or empty).

Convexity: Utility Functions

The characteristic of utility functions that represent convex preferences is **quasi-concavity**.

Definition

A function $u : X \to \mathbb{R}$ is **quasi-concave** if, for every x, y with $u(x) \geq u(y)$ and every $\alpha \in (0, 1)$,

$$u(\alpha x + (1 - \alpha) y) \geq u(y).$$

A function $u : X \to \mathbb{R}$ is **strictly quasi-concave** if, for every x, y with $u(x) \geq u(y)$ and $x \neq y$ and every $\alpha \in (0, 1)$,

$$u(\alpha x + (1 - \alpha) y) > u(y).$$

Exercise: show that u is quasi-concave iff, for every $r \in \mathbb{R}$, the upper contour set $\{x \in X : u(x) \geq r\}$ is convex.

Convexity: Utility Functions

Theorem

Suppose utility function u represents preferences $\succsim$. Then:

$$\begin{aligned} u \text{ is quasi-concave} &\quad\Leftrightarrow\quad \succsim \text{ is convex} \\ u \text{ is strictly quasi-concave} &\quad\Leftrightarrow\quad \succsim \text{ is strictly convex} \end{aligned}$$

Warning: *convex* preferences are represented by quasi-*concave* utility functions.

Convex preferences get that name because they make upper contour sets convex.

Quasi-concave utility functions get that name because quasi-concavity is a weaker property than concavity.

Separability

Often very useful to restrict ways in which a consumer's preferences over one kind of good can depend on consumption of other goods.

If allowed arbitrary interdependencies, would need to observe consumer's entire consumption bundle to infer anything.

Properties of preferences that separation among different kinds of goods are called **separability properties**.

Weak Separability: Preferences

"Preferences over one kind of goods don't depend on what other goods are consumed."

- Let $J_1, \ldots J_m$ be a list of m mutually exclusive subsets of the set of goods.
- Let J_k^C be the complement of J_k.
- Given a vector x, let x_{J_k} be the vector of those goods in J_k.

Definition

$\succsim$ is **weakly separable** in $J_1, \ldots, J_m$ if, for every $k \in 1, \ldots, m$, every $x_{J_k}, x'_{J_k} \in \mathbb{R}^{|J_k|}$, and every $x_{J_k^C}, x'_{J_k^C} \in \mathbb{R}^{\left|J_k^C\right|}$,

$$\left(x_{J_k}, x_{J_k^C}\right) \succsim \left(x'_{J_k}, x_{J_k^C}\right) \Leftrightarrow \left(x_{J_k}, x'_{J_k^C}\right) \succsim \left(x'_{J_k}, x'_{J_k^C}\right)$$

Ex. $X = \{food, clothing, housing\}$, $m = 1$, $J_1 = \{food\}$
$\Longrightarrow$ preferences separable in food, not separable in clothing or housing.

Weak Separability: Utility

Theorem

Suppose utility function u represents preferences $\succsim$. Then $\succsim$ is weakly separable in $J_1, \ldots, J_m$ iff has utility representation of form

$$u(x) = v\left(u_1(x_{J_1}), \ldots, u_m(x_{J_m}), x_{(J_1 \cup \ldots \cup J_m)^C}\right).$$

"Food utility function" u_1, total utility is function of (food utility, clothing, housing).

Other Kinds of Separability

"Strongly separable" preferences imply existence of **additively separable** utility:

$$u(x) = \sum_{k=1}^{m} u_k\left(x_{J_k}\right).$$

"No wealth effects in good 1" imply existence of **quasi-linear** utility:

$$u(x) = x_1 + v\left(x_2, \ldots, x_n\right).$$

Good 1 is called a **numeraire** (or "money").