12.864 Inference from Data and Models   18 February 2003
Problem Set No. 1   Due: 26 February 2003

1. You have a differential equation

$$\frac{d^2x(t)}{dt^2} - k^2 x(t) = 0, \ 0 \le t \le 1,$$

subject to the initial conditions,

$$x(0) = 0, \frac{dx(0)}{dt} = 1.$$

(a) Solve it analytically and explicitly, with $k = 1$.

(b) You are now given *also* a terminal condition $x(1) = a$. For what values of a is there a solution?

(c) Now suppose that both initial and final conditions are missing, but you have instead $x(1/2) = 0.521$; what are the possible values for the two initial conditions that are consistent with this value (ignore the terminal condition).

2. Now consider the same problem as above, but write the differential equation in a discrete form having at least rough accuracy.

(a) Treating the problem as one of solving a set of simultaneous equations, solve the conventional forward problem with two initial conditions $x(0) = 0, x(\Delta t) = \Delta t$. Comment on its resemblance to the solution to (1a).

(b) Let $a = 1.2$, in 1(b). Solve the problem using ordinary least-squares. Solve it *insisting,* to high accuracy, that $x(1) = 1.2$. What is the difference?

(c) With $x(1/2) = 0.521$, find a minimum norm solution for the initial conditions, with the final condition unknown. (Choose a discretization grid so that $t = 1/2$ is a grid point.

(d) Now suppose the end time is infinitely far into the future and let $x(0) = 1$. Can you find a value of $x(\Delta t)$ so that the solution is bounded for all positive times? Is the result sensitive to the exact value? Hint: One way to proceed is to solve the difference equation represented by the finite difference form.

3. Consider the 3 vectors:

$$\mathbf{e}_1 = [.9863, -.5186, 0.3274]^T, \ \mathbf{e}_2 = [0.2341, 0.0215, -1.0039]^T, \mathbf{e}_3 = [-0.9471, -0.3744, -1.1859]^T$$

Are these a spanning set (basis)? Explain.

How about,

$$\mathbf{e}_1 = [.9863, -.5186, 0.3274]^T, \ \mathbf{e}_2 = [0.2341, 0.0215, -1.0039]^T, \mathbf{e}_3 = [1.1034, -0.5079, -0.1746]^T$$

?

Solution:
1. $C_1 e^{kt} + C_2 e^{-kt}$

$$C_1 + C_2 = 0 \tag{1}$$
$$C_1 - C_2 = 1 \tag{2}$$

, Solution is: $\left[C_1 = \frac{1}{2}, C_2 = -\frac{1}{2} \right]$

$x = 1/2 \exp(t) - 1/2 \exp(-t) = x = \frac{1}{2} e^1 - \frac{1}{2} e^{-1} = 1.1752, \frac{1}{2} e^{1/2} - \frac{1}{2} e^{-1/2} : 0.52110$

2. $\Delta t =$

3. First set is independent. Second set has $\mathbf{e}_3 = \mathbf{e}_1 + 1/2 \mathbf{e}_2$. Can find eigenvalues of corresponding matrices, or try expanding

$$\mathbf{e}_3 = a\mathbf{e}_1 + b\mathbf{e}_2 \tag{3}$$

and compare to correct value.