

Turbulence in the Atmosphere and Oceans

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Course description

The course will present the phenomena, theory, and modeling of turbulence in the Earth's oceans and atmosphere. The scope will range from centimeter to planetary scale motions. The regimes of turbulence will include homogeneous isotropic three dimensional turbulence, convection, boundary layer turbulence, internal waves, two dimensional turbulence, quasi-geostrophic turbulence, and planetary scale motions in the ocean and atmosphere. Prerequisites: the mathematics and physics required for admission to the graduate curriculum in the EAPS department, or consent of the instructor.

Course requirements

Class attendance and discussion, bi-weekly homework assignments.

Reference texts

Frisch, "Turbulence: the legacy of Kolmogorov"

Lesieur, "Turbulence in Fluids", 3rd revised edition

McComb, "The physics of turbulence"

Saffman, "Vortex dynamics"

Salmon, "Lectures on geophysical fluid dynamics"

Strogatz, "Nonlinear Dynamics and Chaos: With Applications in Physics, Biology, Chemistry, and Engineering"

Tennekes and Lumley, "A first course in Turbulence"

Chapter 15

Transformed Eulerian Mean in 3D

In this lecture we show that the Transformed Eulerian Mean theory can be extended to three dimensions. We will restrict the analysis to the quasi-geostrophic equations, but progress can be made for primitive equations as well. Most of the work reported here can be found in Plumb (JAS, 1990) and Wardle and Marshall (JPO, 2000).

Consider the average $\bar{\psi}(x, y, z, t)$ of a 3D, QG system. The average can be a temporal average, a low-pass spatial filter or a temporal filter. The averaged QG equations at $O(1)$ are,

$$f\bar{\mathbf{u}}_z = \hat{\mathbf{z}} \times \nabla \bar{b}, \quad \nabla \cdot \bar{\mathbf{u}} = 0, \quad (15.1)$$

$$(15.2)$$

where \mathbf{u} is the horizontal geostrophic velocity. At $O(Ro)$,

$$\bar{\mathbf{u}}_t + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + f_0 \hat{\mathbf{z}} \times \bar{\mathbf{u}}_a = \mathcal{G} - \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'}, \quad (15.3)$$

$$\bar{b}_t + (\bar{\mathbf{u}} \cdot \nabla) \bar{b} + \bar{w}_a N^2 = \mathcal{B} - \nabla \cdot \overline{\mathbf{u}' b'}, \quad (15.4)$$

$$\nabla \cdot \bar{\mathbf{u}}_a = 0. \quad (15.5)$$

15.1 The residual circulation

As for the zonal mean case, we begin by defining the residual circulation with the aim of removing any skew buoyancy flux from the buoyancy budget. Since N^2 does not depend on x and y , the whole flux is skew and can be eliminated with the following definition of residual circulation,

$$\bar{\mathbf{u}}^\dagger = \bar{\mathbf{u}}_a + \nabla \times \Psi, \quad (15.6)$$

where

$$\mathbf{\Psi} = \frac{\hat{\mathbf{z}} \times \overline{\mathbf{u}'b'}}{N^2}. \quad (15.7)$$

This definition leaves the buoyancy budget in the simple form,

$$\bar{b}_t + (\bar{\mathbf{u}} \cdot \nabla) \bar{b} + \bar{w}^\dagger N^2 = \bar{\mathcal{B}}. \quad (15.8)$$

15.2 Transformed momentum budget

The momentum budget must now be written replacing the ageostrophic velocity with the residual velocity,

$$\begin{aligned} \bar{\mathbf{u}}_t + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + f_0 \hat{\mathbf{z}} \times \bar{\mathbf{u}}^\dagger &= \mathcal{G} - \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'} + f \hat{\mathbf{z}} \times (\nabla \times \mathbf{\Psi}) \\ &= \mathcal{G} - \overline{(\mathbf{u}' \cdot \nabla) \mathbf{u}'} + f \hat{\mathbf{z}} \times \frac{\partial}{\partial z} \left(\frac{\overline{\mathbf{u}'b'}}{N^2} \right) - f \nabla \Phi \\ &= \mathcal{G} - \partial_j M_{ji} \end{aligned} \quad (15.9)$$

where the generalized EP fluxes are given by,

$$\mathbf{M} = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \\ M_{zx} & M_{zy} \end{pmatrix} = \begin{pmatrix} \overline{u'^2} + f\Phi & \overline{u'v'} \\ \overline{u'v'} & \overline{v'^2} + f\Phi \\ -fN^{-2}\overline{v'b'} & fN^{-2}\overline{u'b'} \end{pmatrix}. \quad (15.10)$$

The function Φ is yet to be determined (it is arbitrary because of a gauge invariance in the definition of $\mathbf{\Psi}$).

The QGPV flux in 3D is,

$$\overline{u'_i q'_i} = u'_i \overline{\left[v'_x - u'_y + f \frac{\partial}{\partial z} \left(\frac{b'}{N^2} \right) \right]} = \partial_j Q_{ji} \quad (15.11)$$

where,

$$\mathbf{Q} = \begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \\ Q_{zx} & Q_{zy} \end{pmatrix} = \begin{pmatrix} \overline{u'v'} & \overline{v'^2} - \epsilon \\ \epsilon - \overline{u'^2} & -\overline{u'v'} \\ fN^{-2}\overline{u'b'} & fN^{-2}\overline{v'b'} \end{pmatrix}. \quad (15.12)$$

We introduced the energy density,

$$\epsilon = \frac{1}{2} \left(\overline{u'^2} + \overline{v'^2} + N^{-2}\overline{b'^2} \right) = \epsilon_K + \epsilon_P \quad (15.13)$$

given by the sum of the kinetic and potential energy densities.

By choosing,

$$\Phi = \frac{1}{f} (\epsilon - \overline{u'^2} - \overline{v'^2}) = \frac{1}{f} (\epsilon_P - \epsilon_K), \quad (15.14)$$

we have that $\partial_j M_{ji} = \hat{z} \times \partial_j Q_{ji} = \hat{z} \times \overline{\mathbf{u}'q'}$. And the transformed equations become,

$$\bar{\mathbf{u}}_t + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + f_0 \hat{z} \times \bar{\mathbf{u}}^\dagger = \mathcal{G} - \hat{z} \times \overline{\mathbf{u}'q'} \quad (15.15)$$

$$\bar{b}_t + (\bar{\mathbf{u}} \cdot \nabla) \bar{b} + \bar{w}^\dagger N^2 = \bar{\mathcal{B}}, \quad (15.16)$$

$$\nabla \cdot \mathbf{u}^\dagger = 0. \quad (15.17)$$

This system appears to be analogous to the TEM zonal mean problem: there are no explicit eddy terms in the buoyancy budget, and the eddies appear in the momentum budget in the form of a QGPV flux. To be exact as a force per unit mass equal in magnitude, and normal, to the QGPV flux.

15.3 Transformed QGPV budget

The transformed QGPV budget is straightforward, because QGPV is advected only by the horizontal geostrophic flow and not by the ageostrophic velocities (which change upon introduction of the residual velocity),

$$\bar{q}_t + \bar{\mathbf{u}} \cdot \nabla \bar{q} = -\nabla \cdot \overline{\mathbf{u}'q'} + \hat{z} \cdot \nabla \times \mathcal{G} + f \frac{\partial}{\partial z} \left(\frac{\mathcal{B}}{N^2} \right). \quad (15.18)$$

Note that the QGPV budget is only affected by the *divergent part* of the QGPV flux, while the full QGPV flux appears in the residual momentum budget. This is puzzling, because we know that the mean geostrophic circulation can be reconstructed from knowledge of the mean potential vorticity, and hence from a knowledge of the QGPV flux divergence alone. Following Plumb (JAs, 1986) we can show that it is indeed only the divergent part of the QGPV flux that appears in the momentum equation. Let's look at the consequence of subtracting a purely rotational component to the QGPV flux,

$$(\overline{\mathbf{u}'q'})_D = \overline{\mathbf{u}'q'} - \nabla \times \hat{z}\chi. \quad (15.19)$$

The forcing of the momentum equation becomes,

$$-\hat{z} \times \overline{\mathbf{u}'q'} = -\hat{z} \times (\overline{\mathbf{u}'q'})_D + \nabla \chi. \quad (15.20)$$

the term $\nabla \chi$ can then be absorbed into a redefinition of the ageostrophic circulation defining,

$$\Phi = \frac{1}{f} (\epsilon_P - \epsilon_K + \chi). \quad (15.21)$$

Using this technique, we can remove any rotational component of the QGPV flux from the momentum equation.

In conclusion, if we wish to parameterize the eddy forcing, it is only the divergence of the QGPV flux that we need to parameterize in order to calculate the response of the mean geostrophic flow. Notice, however, that a parameterization of the divergent QGPV eddy flux might be elusive, because the QGPV variance budget in 3D is not as useful as in the zonal problem.