

# Homework assignment 3

## Problem 1. Tracer spectra.

Glenn showed that the eddy kinetic energy spectrum for homogeneous isotropic turbulence in two dimensions has different slopes in the direct and inverse cascade ranges. Can you derive the spectral slopes for a passive tracer advected by the two dimensional turbulent flow? Assume that you are in the inertial-convective regime (i.e. molecular friction and diffusion do not matter), but discuss separately the tracer spectrum for the direct and inverse cascade regimes. Tennekes and Lumley's book and Vallis' one are good references for this problem.

## Problem 2. Rotational constraints on 3D turbulence.

Over most of the ocean  $N/f > 1$ , so that turbulence is more likely to be influenced by stratification than by rotation. However, in regions where  $N \approx 0$ , such as the surface mixed layer, rotation may constrain unstratified turbulence. Assuming that rotation constrains the turbulent motions when the Rossby number  $U/fL < 1$ , where  $U$  is the velocity scale, and  $L$  the length scale, find the length scale  $\ell$  above which rotation affects the turbulence. Assume for  $L < \ell$  the turbulent motions are in the inertial range of the Kolmogorov spectrum.

## Problem 3. Transformed Eulerian Mean.

This problem shows how eddies can drive a mean advection with skew eddy fluxes. Let us consider small-amplitude (linear), quasi-geostrophic, Rossby waves in a  $\beta$ -channel  $0 < y < L$ , semi-infinite in the vertical, forced by a zonal mean flow over small amplitude topography  $h'(x, y)$  centered on  $z = 0$ . The bottom boundary is an impermeable surface at  $z = h'(x, y)$ ,

$$h'(x, y) = H_0 \cos(kx) \sin(ly), \quad (1)$$

where  $l = \pi y/L$ . The basic state of the fluid is Boussinesq, with uniform stratification  $N^2$  and a spatially uniform westerly flow  $U$ ; there is no dissipation.

The linearized QGPV equation is,

$$\partial_t q' + U \partial_x q' + v' \partial_y Q = 0, \quad (2)$$

where the mean PV gradient is  $\partial_y Q = \beta$ . The relation between QGPV and the

streamfunction  $\psi$  is,

$$q' = \partial_x^2 \psi' + \partial_y^2 \psi' + \frac{f_0^2}{N^2} \partial_z^2 \psi'. \quad (3)$$

The boundary condition of no-normal flow is,

$$w'|_{z=0} = \partial_t h' + (u' + U) \partial_x h' + v' \partial_y h'. \quad (4)$$

1. Show that the linearized boundary condition at  $z = 0$  is,

$$w'|_{z=0} = U \partial_x h'. \quad (5)$$

2. Solve for  $\psi'$ , that is find solutions to the linear QGPV equation that satisfy the boundary condition. Show that these solutions are vertically propagating Rossby waves.
3. Using your solution for  $\psi'$ , write the corresponding expressions for  $u'$ ,  $v'$ ,  $w'$ , and  $b'$ . Draw schematics of these variables as a function of  $x$  and  $z$ .
4. Compute the Eliassen-Palm fluxes for these waves, that is compute the zonal averages of the momentum and buoyancy fluxes,  $\overline{u'v'}$ ,  $\overline{v'b'}$ , where,

$$\overline{(\ )} = \int_0^{2\pi/k} (\ ) dx. \quad (6)$$

Show that there is an horizontal buoyancy flux, i.e. a an eddy flux along mean density contours (in QG density contours are to leading order horizontal).

5. Compute the residual velocities  $v^\dagger$  and  $w^\dagger$ . Show that there is a non-zero eddy-induced vertical velocity. [Hint: the mean velocity  $\bar{w}$  is not zero.]
6. Explain why this problem is an example of the non-acceleration theorem.