Course 12.812, General Circulation of the Earth's Atmosphere
Prof. Peter Stone
Section 8: Lorenz Energy Cycle

Energy Forms: As we saw in our discussion of the heat budget, the energy content of the atmosphere per unit mass can be expressed as the sum of four terms:

$$E = I + \Phi + LH + K$$
$$I = C_V T = \text{internal energy}$$
$$\Phi = gz = \text{potential energy}$$
$$K = \frac{1}{2}\left(u^2 + v^2 + w^2\right) \cong \frac{1}{2}\left(u^2 + v^2\right) = \text{kinetic energy}$$
$$LH \cong L_V q = \text{latent heat.}$$

We assume C_V and L_V are constants.

If we assume hydrostatic equilibrium, then for the total internal energy in the column of the atmosphere we have:

$$\int_0^\infty \rho I dz = \int_0^\infty \rho C_V T dz = \int_0^{p_s(z=0)} \frac{C_V T}{g} dp;$$ for the potential energy we have

$$\int_0^\infty \rho \Phi dz = \int_0^\infty \rho g z dz = \int_0^{p_s(z=0)} z dp = \cancel{\int_0^{p_s(z=0)} d(zp)}^{\,0} - \int_\infty^0 p dz$$

$$= \int_0^\infty p dz = \int_0^\infty \rho R T dz = \int_0^p \frac{RT}{g} dp$$; (we choose $z = 0$ to be a level below the lowest surface, z_s min, and for convenience, in the presence of topography, we take $p = p_s$ for $z < z_s$. We will see later why. Note that the zero point for potential energy is arbitrary.)

Thus $$\int_0^\infty \rho\left(I + \Phi\right) dz = \int_0^{p_s(z=0)} \frac{\left(C_V + R\right)}{g} dp = \int_0^{p_s(z=0)} \frac{C_P T}{g} dp.$$

Therefore in a column of the atmosphere the sum of I.E. and P.E. equals the enthalpy, and all three are equal within a constant factor. This is also sometimes called the total potential energy = TPE. This can be re-written in terms of the potential temperature,

$$\theta = T\left(p_0/p\right)^\kappa \quad \text{where } \kappa = \frac{R}{C_P} \cong 2/7:$$

$$T dp = \theta\left(\frac{p}{p_0}\right)^\kappa dp = \theta \frac{1}{\left(1+\kappa\right) p_0{}^\kappa} dp^{\kappa+1};$$

$$\theta dp^{\kappa+1} = d\left(\theta p^{\kappa+1}\right) - p^{\kappa+1} d\theta;$$

$\therefore$ integrating by parts we find for the total potential energy:

$$\int_0^{p_s(z=0)} \frac{C_P}{g} T dp = \frac{C_P}{g} \frac{1}{(1+\kappa) p_0^{\kappa}} \left[\theta p^{\kappa+1} \Big|_0^{p_s(z=0)} - \int_{p=0}^{p=p_s, z=0} p^{\kappa+1} d\theta \right]$$

What are the limits on θ? We now adopt the convention that below the surface, $z < z_s$, $\theta < \theta_S =$ surface θ, and $p = p_s =$ constant, and $\theta \to 0$ at $z = 0$. (We implicitly assume that $d\theta / dz > 0$, i.e. the system is always stably stratified.)

$$\therefore \theta p^{\kappa+1} \Big|_0^{p_s} = 0 \text{ and } \frac{C_P}{g(1+\kappa) p_0^{\kappa}} \int_0^{\infty} p^{\kappa+1} d\theta = \text{TPE}$$

However, we note that in a stably stratified atmosphere like ours much of the TPE can never be released or converted to KE. In particular, consider a typical arrangement of the isentropic surfaces as shown in the first diagram below.

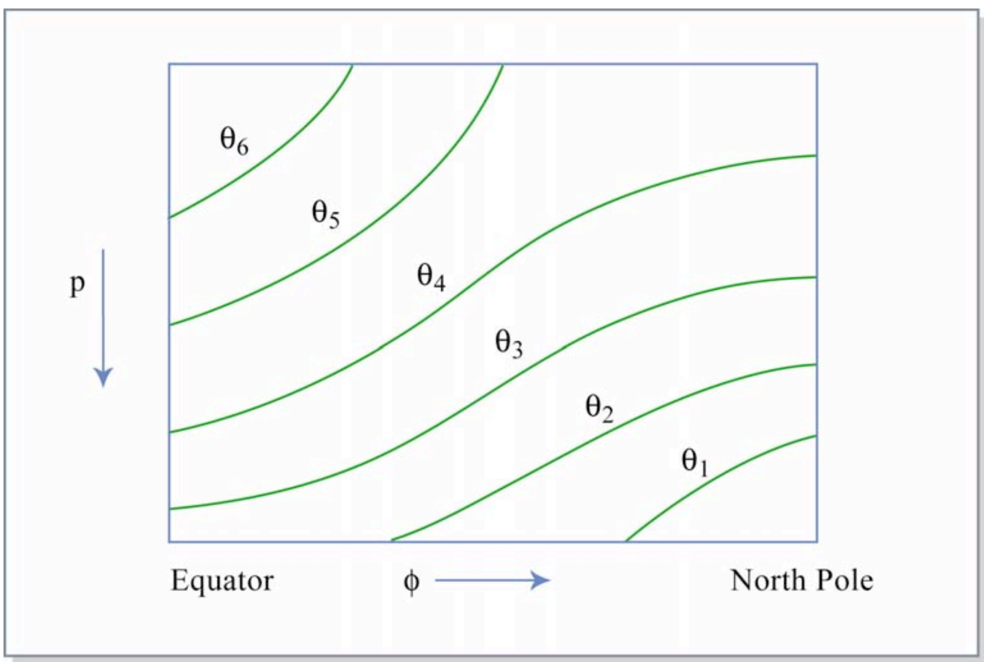


Figure by MIT OCW.

$\theta_6 > \theta_5 ... > \theta_1$. As we see there are horizontal gradients and because of this PE can be released by the motions in spite of the stable stratification. Consider for example a displacement like that shown below.

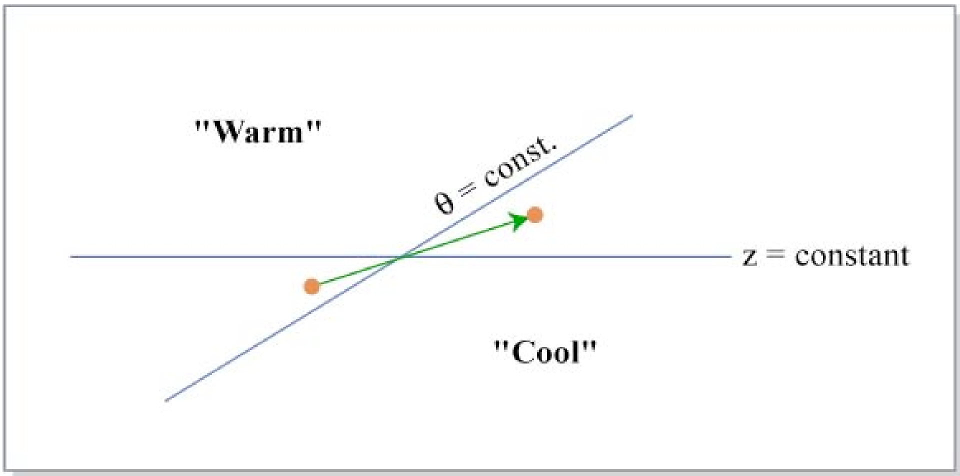


Figure by MIT OCW.

For this displacement high θ air is being introduced into lower θ surroundings, so $\theta' > 0$, but $w' > 0$ also, i.e., $\overline{w'\theta'} > 0$, warm air is rising, and PE is being converted into KE This is of course the mechanism of baroclinic instability.

Now suppose we re-arrange the isentropes adiabatically (so no new energy is added) and while conserving mass, in such a way as to eliminate any horizontal gradients. Since in H.E. p is just the amount of mass above any given level, conserving mass means the integrated or average p above any θ surface is conserved. The average p is $\tilde{p}(\theta) = \frac{1}{A}\int p(x,y,\theta)\,dA$ and using this definition, one can now construct the state with no horizontal gradients, as shown below.

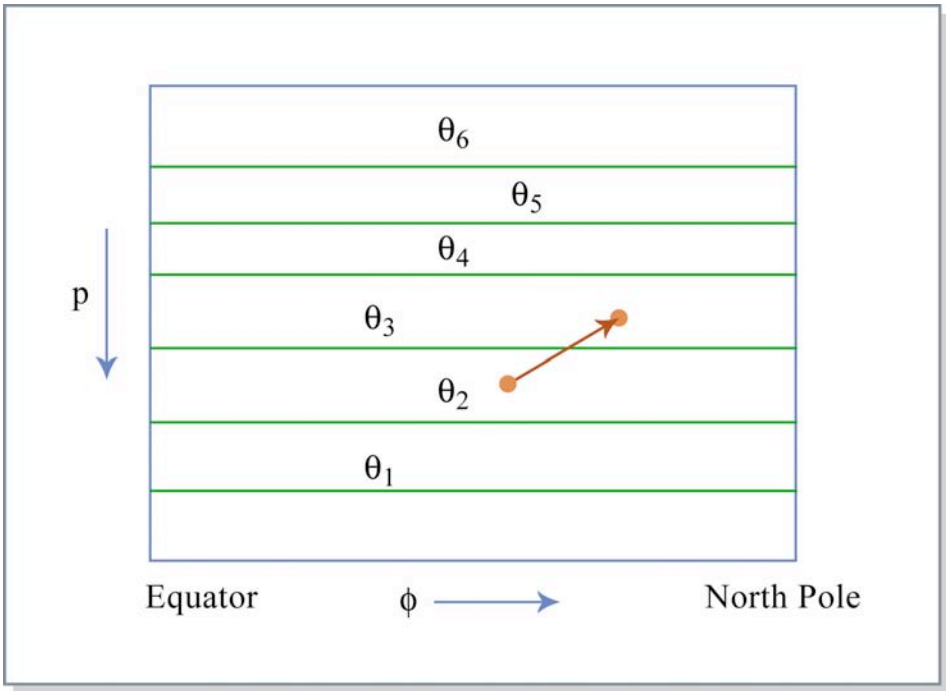


Figure by MIT OCW.

In this new state, since $\partial\theta/\partial z > 0$, there are no displacements which can release PE, because now for any displacement, $\overline{w'\theta'} < 0$. Thus the PE of this state is unavailable.

This led Lorenz (1955) to define the available potential energy as the difference between the TPE in the actual atmosphere and that in the adjusted state just described. If we call this P, then for a given system or region,

$$P = \frac{C_P}{g(1+\kappa)p_0^{\kappa}} \int_A dxdy \int_0^{\infty} d\theta \left(p^{\kappa+1} - \tilde{p}^{\kappa+1}\right)$$

Since $\tilde{p}$ is already integrated over the area, we can rewrite the integral as

$A\int_0^{\infty} d\theta\left(\widetilde{p^{\kappa+1}} - \tilde{p}^{\kappa+1}\right)$. This is the so-called exact formula. Note that, since

$\kappa > 0$, $\left(\widetilde{p^{\kappa+1}}\right) > \left(\tilde{p}\right)^{\kappa+1}$, i.e. P is according to our definition always positive.

Lorenz derived an approximation for P that is commonly used. Let $p = \tilde{p} + p'$, and similarly for other variables:

$$\therefore p^{\kappa+1} = \tilde{p}^{\kappa+1}\left(1+\frac{p'}{\tilde{p}}\right)^{\kappa+1} = \tilde{p}^{\kappa+1}\left(1+\frac{(\kappa+1)p'}{\tilde{p}}+\frac{(\kappa+1)\kappa}{2}+\frac{p'^2}{\tilde{p}^2}+\ldots\right)$$

$$\therefore \widetilde{p^{\kappa+1}} = \tilde{p}^{\kappa+1}\left(1+0+\frac{(\kappa+1)\kappa}{2}\frac{\widetilde{p'^2}}{\tilde{p}^2}+\ldots\right)$$

$$\therefore P \cong \frac{C_p A\cancel{(\kappa+1)}\kappa}{2g\cancel{(1+\kappa)}p_0{}^{\kappa}}\int_0^{\infty}\tilde{p}^{\kappa+1}\left(\frac{\widetilde{p'^2}}{\tilde{p}^2}\right)d\theta = \frac{C_p\kappa}{2gp_0{}^{\kappa}}\int_A dxdy\int_0^{\infty}\tilde{p}^{\kappa-1}p'^2\,d\theta$$

As a test of the accuracy of this approximation, we can take from Oort (1971) the following numbers for $\theta = 300\,\mathrm{K}$ in the annual mean: $p(\text{equator}) = 980\text{mb}$, $p(70\text{N}) = 435\text{mb}, \;\therefore\; \tilde{p} \sim 708\text{mb}, \; |p'| \sim 273\text{mb},$ and the ratio of the 1st neglected term in the series to the first retained term is

$$\frac{(\kappa-1)}{3}\frac{|p'|}{\tilde{p}} = \frac{0.4}{3}\frac{273}{708} = 0.05$$

Carrying out an analysis in θ coordinates is not very convenient, because we would have to interpolate from pressure coordinates, which would introduce error. $\therefore$ Lorenz introduced another approximation. This is based on the approximation that both isentropes and pressure surfaces are quasi-horizontal with very small slopes:

$$\left(\frac{\partial z}{\partial y}\right)_{\theta} = -\frac{\partial\theta/\partial y}{\partial\theta/\partial z} \sim \frac{40^{\circ}/10^4\,\text{km}}{5^{\circ}/\text{km}} \sim 10^{-3} \text{ and}$$

$$\left(\frac{\partial z}{\partial y}\right)_{p} = -\frac{\partial p/\partial y}{\partial p/\partial z} \sim \frac{\rho f u}{\rho g} \sim \frac{10^{-4}10}{10} \sim 10^{-4}.$$

Consider for example an isentrope θ on which p has an average $\tilde{p}$; now consider a point A on $\theta = \text{constant}$. At this point, there will be a pressure deviation p'.

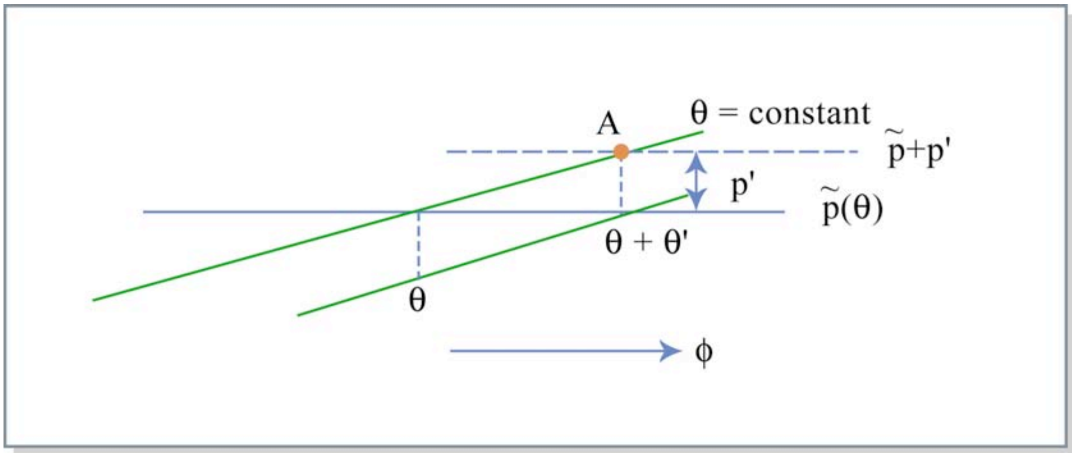


Figure by MIT OCW.

Now consider the θ variations on $\tilde{p}$ at the same latitude, ϕ. Since the θ variations are primarily vertical, or equivalently, vary primarily with pressure, we can write:

$\theta' = \dfrac{-p'}{\partial\tilde{p}/\partial\theta}$. The minus sign is necessary since $\dfrac{\partial\tilde{p}}{\partial\theta} < 0$, (Note that as drawn, $p', \theta' < 0$).

Substituting this into our integrand for P, and changing coordinates from θ to $\tilde{p}$:

$d\theta = \dfrac{d\tilde{p}}{(\partial\tilde{p}/\partial\theta)}$, before integrating over x,y we have that the integrand

$= \dfrac{C_p\kappa}{2gp_0^{\kappa}}\tilde{p}^{\kappa-1}\overline{\theta'^2}\dfrac{\partial\tilde{p}}{\partial\theta}d\tilde{p}$; Note that θ' is now the variation of θ along a constant p surface,

since $\theta = T\left(\dfrac{p_0}{p}\right)^{\kappa}$.

$\therefore \dfrac{\theta'}{\theta} = \dfrac{T'}{T}$; Also $+\dfrac{\partial\theta}{\partial P} = -\dfrac{\kappa}{\Gamma_d}\dfrac{\theta}{p}(\Gamma_d - \Gamma)$ (use H.E.) where $\Gamma_d = g/C_P$.

$\therefore \partial\tilde{p}/\partial\theta = \left(\partial\theta/\partial\tilde{p}\right)^{-1} = \dfrac{\Gamma_d}{\kappa(\Gamma_d - \Gamma)}\dfrac{\tilde{p}}{\theta}$, and $\theta = T\left(\dfrac{p_0}{p}\right)^{\kappa}$;

Substituting this, the integrand simplifies to

$$-\frac{\cancel{C_p}\,\cancel{\kappa}}{2\cancel{g}p_0^{\kappa}}\tilde{p}^{\kappa-1}\frac{\cancel{\Gamma_d}}{\Gamma_d - \Gamma}\frac{\tilde{p}}{\cancel{\kappa\theta}}\frac{\theta^{\cancel{2}}}{T^2}T'^2\,d\tilde{p}$$

$$= -\frac{1}{2}\frac{\cancel{\tilde{p}^{\kappa-1}}}{\cancel{p_0^{\kappa}}}\frac{\cancel{\tilde{p}}}{\Gamma_d - \Gamma}\left(\frac{\cancel{p_0}}{\cancel{\tilde{p}}}\right)^{\kappa}\frac{T'^2}{T}\,d\tilde{p}$$

$= -\frac{1}{2}\frac{T'^2}{T(\Gamma_d - \Gamma)}d\tilde{p};$ Now we substitute into the integration and absorb the minus sign by reversing the limits of integration:

$$\therefore P = \frac{1}{2}\int_A dxdy \int_0^{p_s} d\tilde{p}\frac{T'^2}{T(\Gamma_d - \Gamma)}.$$

Finally Lorenz assumed that the coefficient of T'^2 in the integrand varied slowly compared to T'^2, and replaced T and Γ by their <u>area</u> weighted means $\tilde{T}$ and $\tilde{\Gamma}$. (i.e., 3rd order quantities are again being neglected.) This is not as good an approximation as the earlier ones. (From Oort's data I estimate ~10% error, primarily from T.) Once this assumption is made we can write (noting that now $\tilde{p}$ is a dummy variable)

$$P = \frac{1}{2}\int_0^{p_s} dp \frac{1}{\tilde{T}(\Gamma_d - \Gamma)}\int_A dxdyT'^2.$$

Note that

$$\frac{1}{A}\int dxdyT'^2 = \frac{1}{A}\int dxdy\left(T - \tilde{T}\right)^2$$

$$= \frac{1}{A}\int dxdy\left(T^2 - 2T\tilde{T} + \tilde{T}^2\right)$$

$$= \frac{1}{A}\int dxdy\left(T^2 - \tilde{T}^2\right); \;\; \therefore P = \frac{1}{2}\int_0^{p_s}\frac{dp}{\tilde{T}(\Gamma_d - \tilde{\Gamma})}\int dxdy\left(T^2 - \tilde{T}^2\right)$$

The first term can be expanded in the space domain: $T = [T] + T^*$,

$$[T^2] = [T]^2 + [T^{*2}]$$

$\therefore$ we can decompose $\tilde{P}$ into mean and eddy components; the mean is

$$\boxed{P_M = \frac{1}{2}\int_0^{p_s}\frac{dp}{\tilde{T}(\Gamma_d - \tilde{\Gamma})}\int dxdy\left([T]^2 - \tilde{T}^2\right).}$$

(You may verify that $P_M = \frac{1}{2}\int_A dxdy\int_0^{p_s} dp\frac{[T']^2}{\tilde{T}(\Gamma_d - \tilde{\Gamma})}$.)

We have included the $\tilde{T}^2$ term in the mean available potential energy because it has no zonal variations. The eddy component is the remaining part,

$$\boxed{P_E = \frac{1}{2}\int_0^{p_s}\frac{dp}{\tilde{T}(\Gamma_d - \tilde{\Gamma})}\int [T^{*2}]dydx.}$$

Now let us compare P to TPE:

$$\frac{\frac{1}{2}\int_0^{p_s} dp \frac{1}{\tilde{T}\left(\Gamma_d - \tilde{\Gamma}\right)} \int dxdyT'^2}{\frac{C_P}{g}\int_0^{p_s} dp \int dxdyT} \sim \frac{1}{2}\frac{g}{C_p}\frac{T'^2}{\tilde{T}T\left(\Gamma_d - \tilde{\Gamma}\right)}$$

$$T \sim \tilde{T};\ \frac{g}{C_P} = \Gamma_d;\ T' \sim \Delta T;$$

$$\therefore \frac{P}{TPE} \sim \frac{1}{2}\frac{\left(\Delta T\right)^2}{T^2\left(1 - \frac{\tilde{\Gamma}}{\Gamma_d}\right)};\ \Delta T \sim 30^o,\ T \sim 250^o;\ \tilde{\Gamma} \sim \frac{1}{2}\Gamma_d$$

$\therefore \frac{P}{TPE} \sim \left(\frac{30}{250}\right)^2 \sim \frac{1}{70}$; Hence only ~1 to 2% of the TPE is available for conversion to kinetic energy.

<u>Balance Equations for Kinetic Energy</u>:
We have already derived this for the eddy kinetic energy, integrated over the whole atmosphere, (remember, $K_E = \frac{1}{2}\left(\left[u^{*2}\right] + \left[v^{*2}\right]\right)$), i.e.

$$\frac{\partial}{\partial t}\int K_E dydp = \int g\left(\left[\omega\frac{\partial Z}{\partial p}\right] - \left[\omega\right]\frac{\partial\left[Z\right]}{\partial p}\right)dydp + \int\left(\left[\vec{v}_H \cdot \vec{F}\right] - \left[\vec{v}_H\right]\cdot\left[\vec{F}\right]\right)dydp$$
$$+\int\left[u\right]\left(\frac{\partial}{\partial y}\left[uv\right] + \frac{\partial}{\partial p}\left[u\omega\right]\right)dydp + \int\left[v\right]\left(\frac{\partial}{\partial y}\left[v^2\right] + \frac{\partial}{\partial p}\left[v\omega\right]\right)dydp.$$

Let us expand these terms in the space domain, $u = \left[u\right] + u^*$, etc.

Consider the first term: expanding, and also substituting for H.E.,
$\frac{\partial Z}{\partial p} = -\frac{1}{\rho g} = -\frac{\alpha}{g}$; we have $-\int\left(\left[\omega\alpha\right] - \left[\omega\right]\left[\alpha\right]\right)dydp = -\int\left[\omega^*\alpha^*\right]dydp.$
The term

$$\boxed{C\left(P_E, K_E\right) \equiv -\frac{1}{g}\int\left[\omega^*\alpha^*\right]dydp}$$

represents a conversion of eddy potential energy to eddy kinetic energy, i.e. in pressure coordinates <u>in zonal planes</u>, if warm air is rising $\left(\alpha^* > 0, \omega^* < 0\right)$ and cold air is sinking $\left(\alpha^* < 0, \omega^* > 0\right)$, then $C\left(P_E, K_E\right) > 0$, etc. The factor $\frac{1}{g}$ is necessary to make C have units of energy per unit time. Also, recall that this is <u>total</u> potential energy, i.e. internal energy plus conventional potential energy, not just the latter.

Similarly, the second term when expanded becomes simply

$$\boxed{-\int\left[u^*F_x^*+v^*F_y^*\right]dydp = gD\left(K_E\right),}$$

where $D\left(K_E\right)$ represents the dissipation of eddy kinetic energy due to friction. Since $\vec{v}$ and $\vec{F}$ are negatively correlated, the dissipation, $D(K_E)$ is positive.

Now consider the third term, and expand it:

$$\int\left[u\right]\left(\frac{\partial}{\partial y}\left[uv\right]+\frac{\partial}{\partial p}\left[u\omega\right]\right)dydp$$

$$= \int\left[u\right]\left\{\frac{\partial}{\partial y}\left(\left[u\right]\left[v\right]+\left[u^*v^*\right]\right)+\frac{\partial}{\partial p}\left(\left[u\right]\left[\omega\right]+\left[u^*\omega^*\right]\right)\right\}dydp.$$

The first and third terms can be combined and rewritten: expanding the derivatives, we have:

$$\left[u\right]^2\cancel{\frac{\partial\left[v\right]}{\partial y}}+\left[u\right]\left[v\right]\frac{\partial\left[u\right]}{\partial y}+\left[u\right]^2\cancel{\frac{\partial\left[\omega\right]}{\partial p}}+\left[u\right]\left[\omega\right]\frac{\partial\left[u\right]}{\partial p}$$

$$=\left[v\right]\frac{\partial}{\partial y}\frac{1}{2}\left[u\right]^2+\left[\omega\right]\frac{\partial}{\partial p}\frac{1}{2}\left[u\right]^2=\nabla_2\cdot\left(\vec{v}_2\frac{1}{2}\left[u\right]^2\right),$$

because of continuity. Now we have a pure divergence, and can integrate by Gauss' theorem, and applying the B.C. $\vec{v}\cdot\hat{n}=0$, we get no contribution from these terms. Only the eddy terms survive. The same thing happens with the fourth term in the equation, and these terms are combined to define another conversion, this time a conversion from eddy kinetic energy to mean kinetic energy:

$$\boxed{C\left(K_E,K_M\right)=-\frac{1}{g}\int\left(\left[u\right]\frac{\partial}{\partial y}\left[u^*v^*\right]+\left[u\right]\frac{\partial}{\partial p}\left[u^*\omega^*\right]+\left[v\right]\frac{\partial}{\partial y}\left[v^*\right]^2+\left[v\right]\frac{\partial}{\partial p}\left[v^*\omega^*\right]\right)dydp}$$

Note that in the Q-G approximation, the vertical flux terms would disappear and the $\left[v\right]$ term would be small, of order Ro.

Thus when there is convergence of eddy momentum flux positively correlated with the momentum itself, either zonal or meridional, then $C\left(K_E,K_M\right)>0$. (Peixoto and Oort (1992) give the conversions in the more complicated spherical coordinates.) Thus we can now write for our eddy kinetic energy equation:

$$\boxed{\frac{\partial}{\partial t}\frac{1}{g}\int K_E dydp = C\left(P_E,K_E\right)-C\left(K_E,K_M\right)-D\left(K_E\right)}$$

Also, we now need merely to substitute this into our earlier result for the sum of K_E and K_M to get an equation for $K_M=\frac{1}{2}\left(\left[u\right]^2+\left[v\right]^2\right)$:

$$\frac{\partial}{\partial t}\int K_M dydp = \int g\left[\omega\frac{\partial Z}{\partial p}\right]dydp + \int\left[\vec{v}_H\cdot\vec{F}\right]dydp - \frac{\partial}{\partial t}\int K_E dydp;$$

Rewriting the ω term as before, and substituting for $C(P_E,K_E)$ and $D(K_E)$ we get

$$\frac{\partial}{\partial t}\int K_M dydp = -\int[\omega\alpha]dydp + \int[\omega^*\alpha^*]dydp + \int\left[\vec{v}_H\cdot\vec{F}\right]dydp - \int\left[\vec{v}_H{}^*\cdot\vec{F}^*\right]dydp + gC(K_E,K_M)$$

$$= -\int[\omega][\alpha]dydp + \int\left[\vec{v}_H\right]\cdot\left[\vec{F}\right]dydp + gC(K_E,K_M).$$

Define $\boxed{C(P_M,K_M) = -\frac{1}{g}\int[\omega][\alpha]dydp;}$

i.e. now warm air rising and cold air sinking in <u>meridional planes</u> converts zonal mean total potential energy to zonal mean kinetic energy; and define

$\boxed{D(K_M) = -\frac{1}{g}\int\left[\vec{v}_H\right]\cdot\left[\vec{F}\right]dydp}$ = dissipation of the zonal mean flow, and we have

$$\boxed{\frac{\partial}{\partial t}\frac{1}{g}\int K_M dydp = C(K_E,K_M) + C(P_M,K_M) - D(K_M)}\ .$$

<u>Balance equation for mean available potential energy</u>:

We need an equation for $P_M = \frac{1}{2}\int dxdy\int_0^{P_s} dp\frac{[T']^2}{\tilde{T}(\Gamma_d-\tilde{\Gamma})}$; $T' = T-\tilde{T}$; $\theta' = \theta-\tilde{\theta}$,

$\tilde{T} = \tilde{T}(\ )$, etc. We start by writing the heat balance equation in p coordinates with θ as the dependent variable:

$$C_P\frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} = \dot{H};\ \ \omega = \frac{dp}{dt};\ \ \theta = T\left(\frac{p_0}{p}\right)^{\kappa};\ \ \kappa = \frac{R}{C_P}$$

$$\omega\frac{\partial T}{\partial p} - \frac{\omega}{\rho C_P} = \omega\left[\left(\frac{p}{p_0}\right)^{\kappa}\frac{\partial\theta}{\partial p} + \frac{\kappa}{p}\cancel{\left(\frac{p}{p_0}\right)^{\kappa}}\theta - \frac{RT}{pC_P}\right]$$

Substitute and divide through by $C_P\left(\frac{p}{p_0}\right)^{\kappa} = C_P\frac{T}{\theta}$

$$\therefore \frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + \omega\frac{\partial\theta}{\partial p} = \left(\frac{p_0}{p}\right)^{\kappa}\frac{\dot{H}}{C_P} = \frac{\partial\theta}{\partial t} + \nabla\cdot\vec{v}\theta.$$

Substitute $\theta = \theta' + \tilde{\theta}(p)$; also $\dot{H} = \dot{H}' + \tilde{H}(\ )$

$$\therefore \frac{\partial}{\partial t}\left(\theta' + \tilde{\theta}\right) + \nabla \cdot v_H\left(\theta' + \tilde{\theta}\right) + \frac{\partial}{\partial p}\left(\omega\theta'\right) + \frac{\partial}{\partial p}\left(\omega\tilde{\theta}\right) = \left(\frac{p_0}{p}\right)^{\kappa} \frac{\left(\dot{H}' + \tilde{\dot{H}}\right)}{C_P}.$$

Because of continuity, $\nabla \cdot v_H \tilde{\theta} + \frac{\partial}{\partial p}\left(\omega\tilde{\theta}\right) = \omega\frac{\partial\tilde{\theta}}{\partial p}$. Substituting this into the equation and take the x,y average, $\left(\tilde{\ }\right)$; using the B.C., $\vec{v} \cdot \hat{n} = 0$, we have

$$\frac{\partial\tilde{\theta}}{\partial t} + 0 + \frac{\partial}{\partial p}\left(\widetilde{\omega\theta'}\right) + \cancel{\tilde{\omega}}^{0}\frac{\partial\tilde{\theta}}{\partial p} = \left(\frac{p_0}{p}\right)^{K}\frac{\tilde{\dot{H}}}{C_P};\ \tilde{\omega} = 0 \text{ because of continuity.}$$

Note that this is precisely the equation on which global mean radiative-convective models are based (cf. Hantel, 1976). Also, this equation in effect tells us that in equilibrium, the mean diabatic heating is balanced by the mean eddy vertical heat flux, $\widetilde{\omega\theta'}$. Now subtract this equation from the $\theta' + \tilde{\theta}$ equation:

$$\therefore \frac{\partial\theta'}{\partial t} + \nabla \cdot \vec{v}_H\theta' + \frac{\partial}{\partial p}\left(\omega\theta' - \widetilde{\omega\theta'}\right) + \omega\frac{\partial\tilde{\theta}}{\partial p} = \left(\frac{p_0}{p}\right)^{\kappa}\frac{\dot{H}'}{C_P}$$

Take the zonal mean of this equation, and multiply by $G\left[\theta'\right]$ where

$$\boxed{G = \left(\frac{p}{p_0}\right)^{2\kappa}\frac{1}{\tilde{T}\left(\Gamma_d - \tilde{\Gamma}\right)};}$$

$$\therefore \underset{\text{I}}{G\frac{\partial}{\partial t}\frac{1}{2}\left[\theta'\right]^2} + \underset{\text{II}}{G\left[\theta'\right]\frac{\partial}{\partial y}\left[v\theta'\right]} + G\left[\theta'\right]\frac{\partial}{\partial p}\Big(\underset{\text{III}}{\left[\omega\theta'\right]} - \underset{\text{IV}}{\widetilde{\omega\theta'}}\Big) + \underset{\text{V}}{G\left[\theta'\right]\left[\omega\right]\frac{\partial\tilde{\theta}}{\partial p}} = \underset{\text{VI}}{\frac{G}{C_P}\left(\frac{p_0}{p}\right)^{\kappa}\left[\dot{H}'\right]\left[\theta'\right]}$$

Now consider what will happens to each of these terms when integrated over y,p. (The integration over x is implicit.) In I, in order to obtain the desired term, $\partial P_M/\partial t$, we have to take G inside the time derivative, i.e., we have to neglect the time variations in $\tilde{T}$ and $\tilde{\Gamma}$ in this term. Since these are comparable in magnitude to the already neglected y variations, it is no worse an assumption. If we are doing a global analysis, it is a much better approximation because the seasonal changes in the two hemispheres counteract each other. (See Oort & Peixoto, 1983, Fig. 47). Substituting for G and $\left[\theta'\right]^2$ the I term becomes:

$$\frac{\partial}{\partial t}\int \cancel{\left(\frac{p}{p_0}\right)^{2\kappa}}\frac{1/2}{\tilde{T}\left(\Gamma_d - \tilde{\Gamma}\right)}\cancel{\left(\frac{p_0}{p}\right)^{2\kappa}}\left[T'\right]^2 dydp = \frac{\partial P_M}{\partial t}.$$

Also note that this approximation does not affect the calculation of the mean balance from this equation.

Terms II and III can be combined and simplified by integrating the perfect derivatives, as follows:

$$\int dydpG\left[\theta'\right]\left\{\frac{\partial}{\partial y}\left[v\theta'\right]+\frac{\partial}{\partial p}\left[\omega\theta'\right]\right\}$$

$$= \int dydp\left\{\frac{\partial}{\partial y}\left(\cancel{\left[v\theta'\right]}^{0}G\left[\theta'\right]\right)+\frac{\partial}{\partial p}\left(G\cancel{\left[\theta'\right]}^{0}\left[\theta'\omega\right]\right)-\left[v\theta'\right]\frac{\partial}{\partial y}\left(G\left[\theta'\right]\right)-\left[\omega\theta'\right]\frac{\partial}{\partial p}\left(G\left[\theta'\right]\right)\right\}$$

$$=-\int dydp\left\{\left(\left[v\right]\left[\theta'\right]+\left[v*\theta'*\right]\right)\frac{\partial}{\partial y}\left(G\left[\theta'\right]\right)+\left(\left[\omega\right]\left[\theta'\right]+\left[\omega*\theta'*\right]\frac{\partial}{\partial p}\left(G\left[\theta'\right]\right)\right)\right\}$$

The first and third terms can be rewritten as

$$\left[v\right]\frac{\partial}{\partial y}\frac{1}{2}G\left[\theta'\right]^2+\left[\omega\right]\frac{\partial}{\partial p}\frac{1}{2}G\left[\theta'\right]^2=\nabla_2\cdot\left[\vec{v}_2\right]\frac{1}{2}G\left[\theta'\right]^2,$$

because of continuity, and this integrates to zero. Noting that $\theta^*=\theta'^*$, we now have $\mathrm{II}+\mathrm{III}=-\int dydp\left\{\left[v*\theta*\right]\frac{\partial}{\partial y}\left(G\left[\theta'\right]\right)+\left[\omega*\theta*\right]\frac{\partial}{\partial p}\left(G\left[\theta'\right]\right)\right\}$ and substituting for θ', $\theta*$ and G, this becomes

$$\boxed{-\int dydp\left\{\left[v*T*\right]\frac{\partial}{\partial y}\frac{\left[T'\right]}{\tilde{T}\left(\Gamma_d-\tilde{\Gamma}\right)}+\frac{\left[\omega*T*\right]}{p^\kappa}\frac{\partial}{\partial p}\frac{p^\kappa\left[T'\right]}{\tilde{T}\left(\Gamma_d-\Gamma\right)}\right\}=C\left(P_M,P_E\right).}$$

This represents the conversion from mean to eddy available potential energy. Thus if there are negative correlations between the eddy transports and the gradients of $\left[T'\right]$, (actually in the $\left[\omega*T*\right]$ case it is not the gradient of $\left[T'\right]$, but the coefficient of $\left[T'\right]$ does not vary strongly in p), i.e. if the transports are down gradient, then $C\left(P_M,P_E\right)>0$ and P_M is converted to P_E (see diagram below). In effect, longitudinal T gradients are being created from the meridional and vertical gradients.

Note that in the quasi geostrophic approximation, the vertical eddy flux term would not appear.

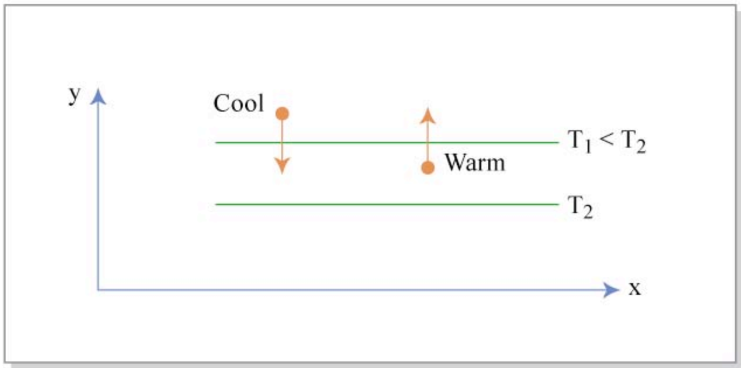


Figure by MIT OCW.

Next consider the term IV: $\iint dydpG\left[\theta'\right]\frac{\partial\left(\widetilde{\omega\theta'}\right)}{\partial p}$

= 0, because $\widetilde{\omega\theta'}$ and G are independent of y, $\widetilde{\theta}'=0$ by definition, and $\therefore\left[\theta'\right]$ integrates to zero over y.

Now consider the term V:

$$V=\int dydpG\left[\theta'\right]\left[\omega\right]\frac{\partial\tilde{\theta}}{\partial p}; \text{ substitute for G, } \theta' \text{ and } \frac{\partial\tilde{\theta}}{\partial p}=\frac{\kappa\left(\tilde{\Gamma}-\Gamma_d\right)}{\Gamma_d}\frac{\tilde{\theta}}{p}. \text{ We find}$$

$$V=-\frac{1}{g}\int dydp\left[\omega\right]\left[\alpha'\right]=+C\left(P_M,K_M\right)$$

Note that $\left[\alpha'\right]$ and $\left[\alpha\right]$ are interchangeable in the integrand because the integral of ω is 0.

The factor of $\frac{1}{g}$ comes in because we are integrating in pressure coordinates.

Note that $\alpha=\frac{1}{\rho}=\frac{RT}{p}$; $\therefore\alpha'=\frac{RT'}{p}$; $\therefore$ if warm air rises and cold air sinks, $\left[\omega\right]$ and $\left[\alpha'\right]$ in meridional planes are negatively correlated, and P_M is converted to K_M. Also note that we can replace $\left[\alpha'\right]$ by $\left[\alpha\right]$ because $\left[\omega\right]\left[\tilde{\alpha}\right]$ integrates to zero.

Finally, the VI term, upon substituting for G and θ', becomes:

$$VI=\int\frac{G}{C_p}\left(\frac{p_0}{p}\right)^{K}\left[\dot{H}'\right]\left[\theta'\right]dydp$$

$$= \int dydp \frac{[\dot{H}'][T']}{C_p \tilde{T}(\Gamma_d - \tilde{\Gamma})} \equiv G(P_M)$$

$\therefore$ mean available potential energy is generated if the zonal mean T and $\dot{H}$ fluctuation are correlated, i.e. if we have heating where T is high and cooling where T is low. Note that $[T']$ can again be replaced by $[T]$ because $\overline{[\dot{H}'][\tilde{T}]} = 0$.

Now we put it all together to get the P_M equation:

$$\frac{\partial P_M}{\partial t} = G(P_M) - C(P_{M,} P_E) - C(P_{M,} K_M).$$

Finally, a P_E equation can be derived in similar fashion by deriving an equation for $[\overline{T} *^2]$ from the same thermodynamic equation. The already defined conversions for $C(P_E, K_E)$ and $C(P_M, P_E)$ will appear again, and the only new term is a generation term exactly analogous to $G(P_M)$, i.e.

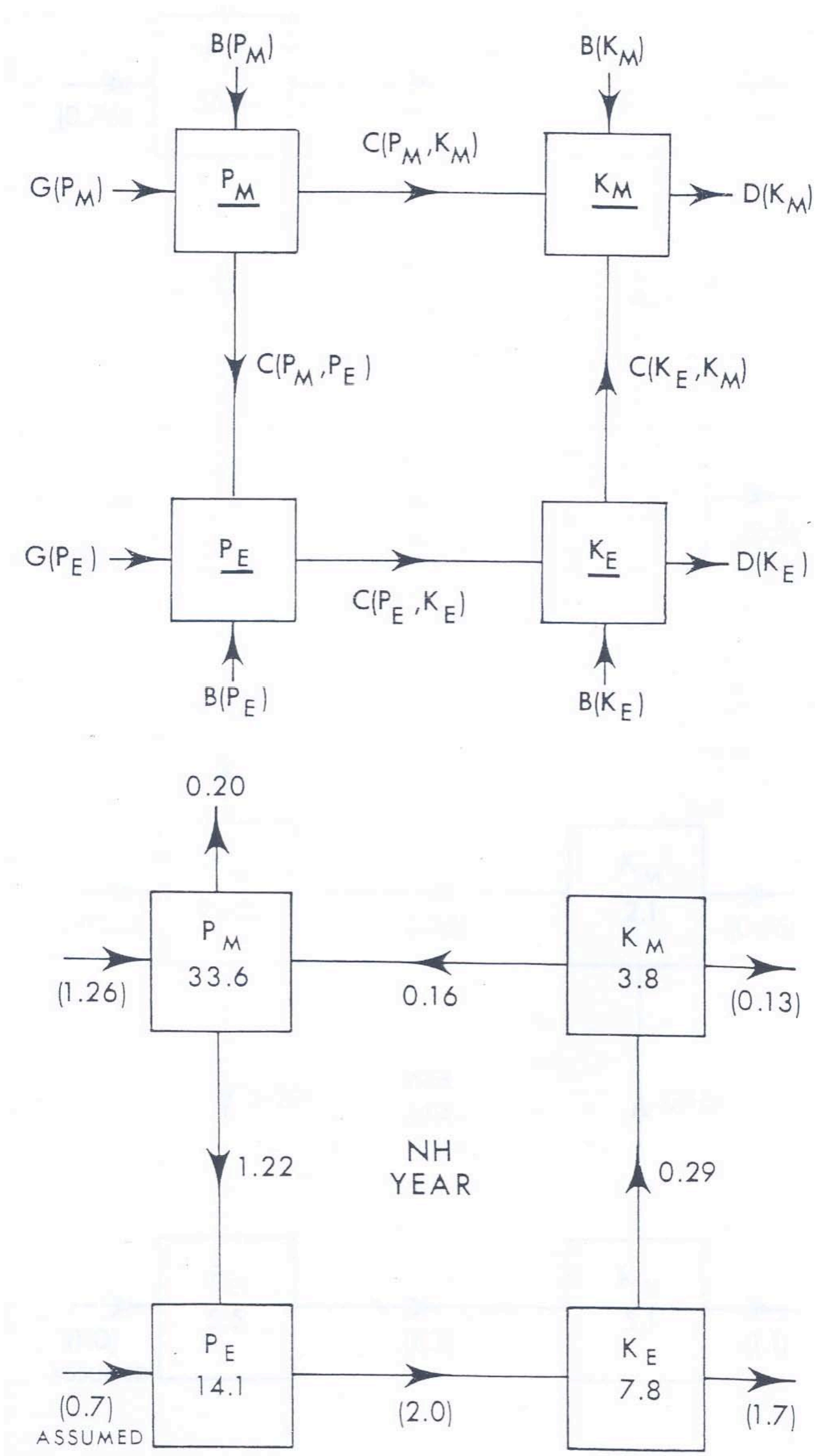
$$G(P_E) = \int dydp \frac{[\dot{H} * T *]}{C_p \tilde{T})(\Gamma_d - \Gamma)}$$

Thus our equation is

$$\frac{\partial P_E}{\partial t} = G(P_E) + C(P_{M,} P_E) - C(P_E, K_E).$$

<u>Observational Analyses</u>: We shall follow Oort & Peixoto (1983). Oort & Rasmussen (1992) cite Oort and Peixoto (1983) as their source, but there are minor discrepancies. We will only look at the Northern Hemisphere analyses, because of the better data base. Note however that if we do, then in general there are boundary fluxes across the equator which must be added to the equations above. Oort & Peixoto (1974) calculated these and found that they were generally not important except for the flux of P_M in the seasonal extremes, $B(P_M)$, associated with the strong seasonal winter Hadley Cells and the term $[v]\frac{1}{2}G[\theta']^2$. Thus we add $B(P_M)$ to the P_M equation, $\frac{\partial P_M}{\partial t} = B(P_M) + \ldots$.

The analyses are based on the 10 years in the MIT general circulation library, and they calculated the four energies, $C(P_M, K_M)$, $C(P_M, P_E)$, and $C(K_E, K_M)$. Note that in evaluating the first two conversions they assumed $Ro \ll 1$, and only retained the leading terms. $G(P_M)$ and $G(P_E)$ cannot be calculated because of lack of data on diabatic heating, $C(P_E, K_E)$ cannot be calculated because of the lack of data on $[\omega^* \alpha^*]$, and $D(K_M)$ and $D(K_E)$ because of lack of data on friction. However if any one of the three $G(P_E)$, $C(P_E, K_E)$ and $D(K_E)$ were known, all the other unknowns can be calculated as residuals, assuming a stationary state (see Fig.1, top). Oort & Peixoto (1983) assumed values for $G(P_E)$ and calculated the others for the annual mean and for the December-January-February and June-July-August seasons. They did not explain the basis for their assumed $G(P_E)$ values in any of their papers. We can regard them as educated guesses. They assumed positive values, which could be explained as being due to a positive correlation between $\dot{H}^*$ and $\dot{T}^*$ (i.e. in zonal planes) due to baroclinic eddies releasing latent heat (which is contained in $\dot{H}$ in the energy equations), i.e. warm air moving poleward rises, leading to condensational heating.



B(P_M)
B(K_M)
C(P_M, K_M)
G(P_M)
P_M
K_M
D(K_M)
C(P_M, P_E)
C(K_E, K_M)
G(P_E)
P_E
K_E
D(K_E)
C(P_E, K_E)
B(P_E)
B(K_E)
0.20
P_M
33.6
K_M
3.8
(1.26)
0.16
(0.13)
1.22
NH
YEAR
0.29
P_E
14.1
K_E
7.8
(0.7)
ASSUMED
(2.0)
(1.7)

Figure 1

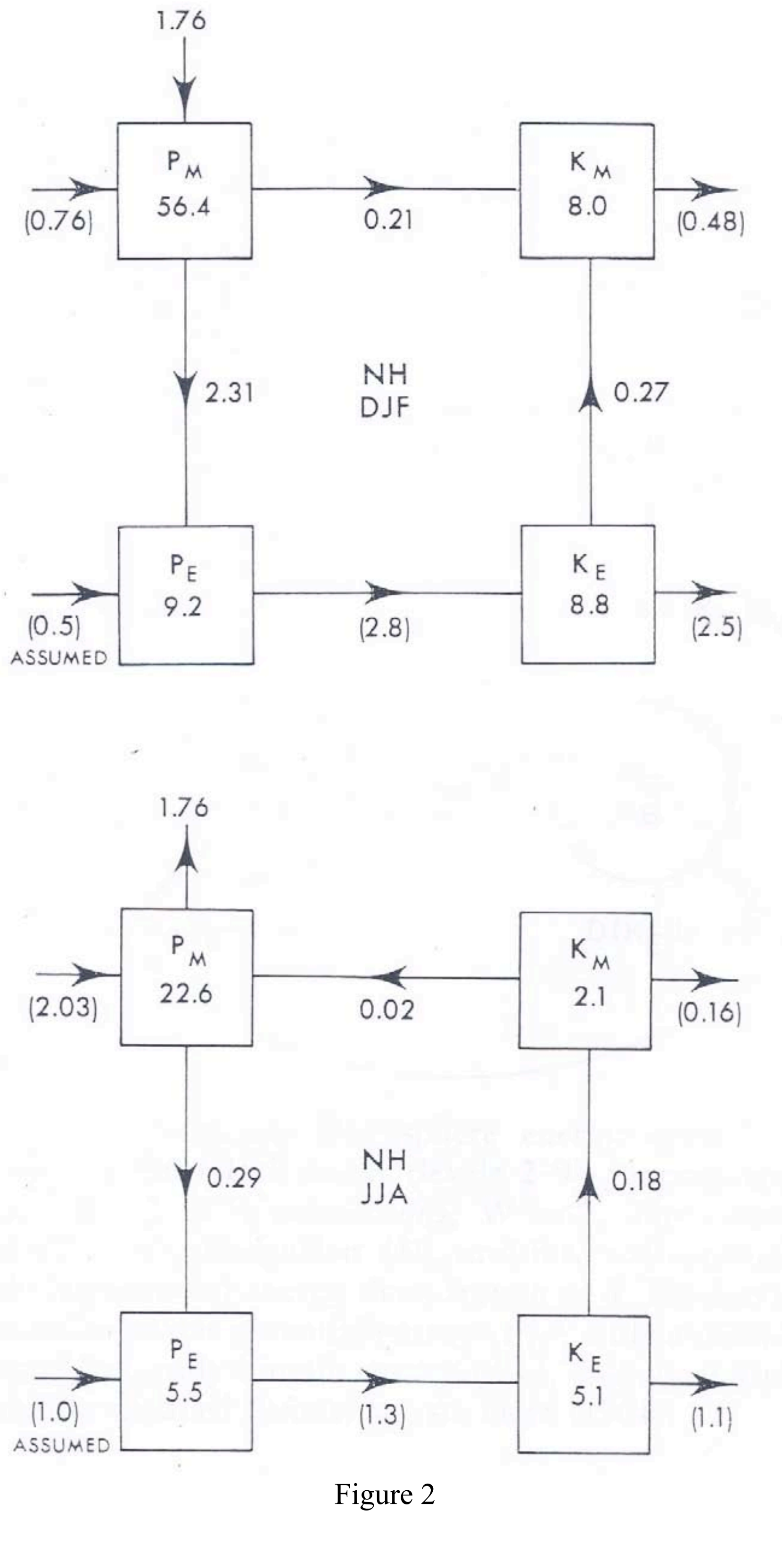

1.76
P_M
56.4
(0.76)
0.21
K_M
8.0
(0.48)
2.31
NH
DJF
0.27
P_E
9.2
(0.5)
ASSUMED
(2.8)
K_E
8.8
(2.5)
1.76
P_M
22.6
(2.03)
0.02
K_M
2.1
(0.16)
0.29
NH
JJA
0.18
P_E
5.5
(1.0)
ASSUMED
(1.3)
K_E
5.1
(1.1)

Figure 2

Fig. 1 (bottom) shows the annual mean Northern Hemisphere energy cycle, and Fig. 2 shows the Northern Hemisphere seasonal cycles. (Note apparent small errors in the residual calculations in the annual mean.) Looking first at the annual mean, we note:

1. The main input is due to $G(P_M)$, caused by heating in warmer low latitudes and cooling in cooler high latitudes. The radiative fluxes cause this both directly and indirectly by driving low latitude moist convection and condensation.
2. From P_M energy flows mainly to P_E, because of baroclinic eddies – i.e., their downgradient heat transport.
3. P_E energy flows mainly to K_E, i.e. K_E is generated by upward eddy heat fluxes, i.e. $[\omega^* T^*] > 0$.
4. This flow is mainly balanced by dissipation, i.e., mainly by friction near the surface.
5. There is however some flow of energy from K_E to K_M, i.e., there is an upgradient transport of momentum by the eddies, on balance. In fact this is the main source of K_M.
6. There is some flow of energy from K_M to P_M, i.e., on balance $[\omega][T] < 0$, i.e. on balance this conversion is dominated by the indirect Ferrel Cell.
7. $D(K_E)$ and $G(P_E)$ are relatively small.

Fig. 2 shows the seasonal changes. Note:

1. In winter the dominant energy source is $B(P_M)$, i.e. P_M is mainly being generated in the Southern Hemisphere, but a lot of it is being transported across the equator to the Northern Hemisphere.
2. $G(P_M)$ is greater in June-July-August, presumably because tropical moist convection and the associated LH release is greater then.
3. The eddy convergences are all much weaker in summer and thus the flow of energy is weaker in summer, and all the energy forms are less.
4. The Ferrel Cell does not dominate $C(P_M, K_M)$ in December-January-February, because it has to contend with the very strong winter Hadley cell.

Because of the difficulty in calculating the energy cycle (calculating G repeatedly for example) there have been very few attempts to calculate it from models. The only one I know of was with an early version of the GISS GCM (Somerville et al, 1974). It used $4°(\text{lat}) \text{x} 5°(\text{long})$ resolution with 9 vertical levels, and a single January was simulated. No boundary fluxes were calculated; the Northern Hemisphere simulated energy cycle is shown in Fig. 12a in Somerville et al (1974). Comparing with the Peixoto and Oort winter results, we note

1. The energy cycles are qualitatively similar.
2. Most of the model's energies are too high (except for K_E)

3. Most of the conversions, generation and dissipation terms are too weak, except for $C\left(P_M, P_E\right)$
4. Gratifyingly, $G\left(P_E\right)$ is 0.6, close to Oort and Peixoto's assumed value.

Other features for the GISS simulation indicate that the differences are not due to the resolution or short time base. The large K_M may be in part because the tropospheric jet is not closed.

Note that the results for the Energy Cycle tell us that the EP theorem is not at all satisfied in the trosposhere, because there is a net exchange of energy between the eddies and the mean flow. $C\left(P_M, P_E\right)$ is about four times $C\left(K_E, K_M\right)$