Laplace Tidal Equations

Basic equations

$$\frac{\partial}{\partial t}\mathbf{u} + (2\Omega + \boldsymbol{\zeta}) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla(\phi + \frac{1}{2}\mathbf{u}\cdot\mathbf{u})$$
$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{u}) = 0$$
$$\frac{D}{Dt}\rho - \frac{1}{c_s^2}\frac{D}{Dt}p = 0$$

The hydrostatic state corresponds to

$$\frac{1}{\overline{\rho}}\nabla\overline{p} = -\nabla\phi \quad \Rightarrow \quad \overline{p} = \overline{p}(\phi/g) \;\; , \quad \overline{\rho} = \overline{\rho}(\phi/g) \;\; with \quad \nabla\phi = g\hat{\mathbf{k}}$$

Expanding the right-hand side pressure and gravitational potential terms using $p \equiv \overline{p} + \overline{\rho}P'$ and $\rho = \overline{\rho} + \rho'$ gives

$$-\frac{1}{\rho}\nabla p - \nabla\phi = \frac{1}{\overline{\rho}+\rho'}\overline{\rho}\nabla\phi - \frac{1}{\overline{\rho}+\rho'}\nabla\overline{\rho}P' - \frac{\overline{\rho}+\rho'}{\overline{\rho}+\rho'}\nabla\phi = -\frac{\overline{\rho}}{\overline{\rho}+\rho'}\nabla P' - P'\frac{\nabla\overline{\rho}}{\overline{\rho}+\rho'} - \frac{\rho'}{\overline{\rho}+\rho'}\nabla\phi$$

Consistent with the linearization to come, we will now keep only first order in perturbation variables

$$-\frac{1}{\rho}\nabla p - \nabla\phi \simeq -\nabla P' - P'\frac{\nabla\overline{\rho}}{\overline{\rho}} - \frac{\rho'}{\overline{\rho}}\nabla\phi = -\nabla P' + b'\hat{\mathbf{k}}$$

where the buoyancy perturbations are given by

$$b' = -\frac{g\rho'}{\overline{\rho}} - \frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z}P'$$

Thus we get the linearized momentum equations

$$\frac{\partial}{\partial t}\mathbf{u}' + 2\Omega\times\mathbf{u}' = -\nabla P' + b'\hat{\mathbf{k}} \tag{1}$$

The mass equation linearizes to

$$\frac{\partial}{\partial t}\frac{\rho'}{\overline{\rho}} + \frac{1}{\overline{\rho}}\nabla\cdot(\overline{\rho}\mathbf{u}') = 0$$

or

$$-\frac{\partial}{\partial t}\frac{b'}{g} - \frac{1}{g}\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z}\frac{\partial}{\partial t}P' + \frac{1}{\overline{\rho}}\nabla\cdot(\overline{\rho}\mathbf{u}') = 0 \tag{2}$$

Finally, the thermodynamic equation gives

$$\frac{\partial}{\partial t}(\rho' - \frac{\overline{\rho}P'}{\overline{c}_s{}^2}) + w'(\frac{\partial\overline{\rho}}{\partial z} + \frac{g\overline{\rho}}{\overline{c}_s{}^2}) = 0$$

If we multiply by $-g/\overline{\rho}$, we find

$$\frac{\partial}{\partial t}(b' - \frac{N^2}{g}P') + w'N^2 = 0 \tag{3}$$

with the Brunt-Väisälä frequency given by

$$N^2 = -\frac{g}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z} - \frac{g^2}{\overline{c}_s{}^2}$$

Geometric approximation

For a thin shell, we can replace Ω by $f\hat{\mathbf{k}}$ and maintain energetic consistency. This may break down near the equator; elsewhere the parts of the Coriolis force associated with the local horizontal component of rotation are negligible. Thus we shall work with (1-3) in the form

$$\frac{\partial}{\partial t}\mathbf{u}' + f\hat{\mathbf{k}} \times \mathbf{u}' = -\nabla P' + b'\hat{\mathbf{k}}$$

$$-\frac{\partial}{\partial t}\frac{b'}{g} - \frac{1}{g}\frac{1}{\overline{\rho}}\frac{\partial \overline{\rho}}{\partial z}\frac{\partial}{\partial t}P' + \frac{1}{\overline{\rho}}\nabla \cdot (\overline{\rho}\mathbf{u}') = 0$$

$$\frac{\partial}{\partial t}(b' - \frac{N^2}{g}P') + w'N^2 = 0$$

We shall also ignore the derivatives of radius in the metric terms so that

$$\frac{1}{r} \to \frac{1}{a}$$

where a is the planetary radius, and terms such as

$$\frac{1}{\overline{\rho}r^2}\frac{\partial}{\partial r}\overline{\rho}r^2w' = \frac{1}{\overline{\rho}(a+z)^2}\frac{\partial}{\partial z}\overline{\rho}(a+z)^2w' \to \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w'$$

Therefore, we can split the horizontal and vertical parts out

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla P \qquad (f.1)$$

$$\frac{\partial}{\partial t}w = -\frac{\partial}{\partial z}P + b \qquad (f.2)$$

$$-\frac{\partial}{\partial t}\frac{b}{g} - \frac{1}{g}\frac{1}{\overline{\rho}}\frac{\partial \overline{\rho}}{\partial z}\frac{\partial}{\partial t}P + \nabla \cdot \mathbf{u} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}(\overline{\rho}w) = 0 \qquad (f.3)$$

$$\frac{\partial}{\partial t}(b - \frac{N^2}{g}P) + wN^2 = 0 \qquad (f.4)$$

where $\mathbf{u}$ is the horizontal velocity (along geopotential surfaces) $(\mathbf{u}' - w'\hat{\mathbf{k}})$ and the gradient and divergence are likewise horizontal operators with no dependence on z. We've dropped all the primes on the wave quantities.

Hydrostatic case

When the motions are hydrostatic,

$$b = \frac{\partial P}{\partial z}$$

and

$$w = \left[\frac{1}{g} - \frac{1}{N^2}\frac{\partial}{\partial z}\right]\frac{\partial P}{\partial t}$$

When we substitute this into the mass equation, we find the

$$-\frac{1}{g}P_{zt} - \frac{1}{g}\frac{\overline{\rho}_z}{\overline{\rho}}P_t + \frac{1}{g}\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}(\overline{\rho}P_t) + \nabla\cdot\mathbf{u} - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}P_t = 0$$

so that

$$\nabla\cdot\mathbf{u} - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}P_t = 0$$

Since the horizontal equations have no coefficients depending on z, we can separate variables

$$\mathbf{u} = \mathbf{u}(x,y,t)F(z) \quad , \quad P = P(x,y,t)F(z)$$

and still have

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{k}}\times\mathbf{u} = -\nabla P \qquad (h.1)$$

In the mass conservation/ thermodynamic eqn., we now have

$$\nabla\cdot\mathbf{u}F(z) - \frac{\partial P}{\partial t}\left[\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}F(z)\right] = 0$$

which will hold when

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}F = -\frac{1}{gH_{eq}}F \qquad (h.2)$$

where we've introduced a notation for the separation constant which makes the horizontal part

$$\frac{\partial}{\partial t}P + gH_{eq}\nabla\cdot\mathbf{u} = 0 \qquad (h.3)$$

look very familiar: equations (h.1) and (h.3) are just the shallow-water equations with an "equivalent depth" H_{eq}.

If we have solid boundaries at $z = 0,\ H$, then $w = 0$ which implies

$$\frac{\partial}{\partial z}F = \frac{N^2}{g}F \quad at \quad z = 0,\ H \qquad (h.4 - fixed)$$

A free surface, on the other hand has $w = \frac{\partial}{\partial t}\eta$ with $\overline{P}(\eta) + \overline{\rho}(0)P(0) = 0 \ \Rightarrow\ P(0) = g\eta$ so that

$$\left[\frac{1}{g} - \frac{1}{N^2}\frac{\partial}{\partial z}\right]\frac{\partial P}{\partial t} = \frac{1}{g}\frac{\partial P}{\partial t} \quad \Rightarrow \quad \frac{\partial}{\partial z}F = 0 \qquad (h.4 - free)$$

Equations (h.2) and (h.4) give a Sturm-Liouville problem with a discrete set of eigenvalues H_{eq} (at least for the system with two boundaries).

Non-hydrostatic case

Let us now separate the vertical and horizontal parts of the full equations (f1-4). To do this, we need to assume a single frequency so that we can solve for w and b in terms of P. From the thermodynamic and vertical momentum equations, we find

$$\frac{\partial^2 w}{\partial t^2} + N^2 w = -P_{zt} + \frac{N^2}{g} P_t \quad \Rightarrow \quad w = \left[\frac{N^2}{g(N^2 - \omega^2)} - \frac{1}{N^2 - \omega^2} \frac{\partial}{\partial z} \right] P_t$$

which reduces to the hydrostatic case when $\omega^2 << N^2$. Likewise the buoyancy satisfies

$$b = \frac{N^2}{N^2 - \omega^2} \left[\frac{\partial}{\partial z} - \frac{\omega^2}{g} \right] P$$

With these forms, the conservation of mass equation looks like

$$\left[-\frac{1}{g} \frac{N^2}{N^2 - \omega^2} \left(\frac{\partial}{\partial z} - \frac{\omega^2}{g} \right) - \frac{1}{g} \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial z} + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho} N^2}{g(N^2 - \omega^2)} - \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho}}{N^2 - \omega^2} \frac{\partial}{\partial z} \right] \frac{\partial P}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

or

$$\left[\frac{1}{g} \frac{\partial}{\partial z} \left(\frac{\omega^2}{N^2 - \omega^2} \right) - \frac{\omega^2}{\overline{c}_s{}^2 (N^2 - \omega^2)} - \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho}}{N^2 - \omega^2} \frac{\partial}{\partial z} \right] \frac{\partial P}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

The horizontal velocities and dynamic pressure P can still have the form $\mathbf{u} = \mathbf{u}(x, y, t) F(z)$, $P = P(x, y, t) F(z)$. But now the vertical structure equation becomes

$$\left[\frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho}}{N^2 - \omega^2} \frac{\partial}{\partial z} - \frac{1}{g} \frac{\partial}{\partial z} \left(\frac{\omega^2}{N^2 - \omega^2} \right) + \frac{\omega^2}{\overline{c}_s{}^2 (N^2 - \omega^2)} \right] F = -\frac{1}{g H_{eq}} F \qquad (VSE)$$

and the separation constant depends on the wave frequency. The boundary conditions become

$$\left[\frac{\partial}{\partial z} - \frac{N^2}{g} \right] F = 0 \qquad (Solid\ B)$$

or

$$\left[\frac{\partial}{\partial z} - \frac{\omega^2}{g} \right] F = 0 \qquad (Free\ B)$$

The horizontal structures still satisfy the Laplace tidal equations

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u} + f \hat{\mathbf{k}} \times \mathbf{u} &= -\nabla P \\ \frac{\partial}{\partial t} P + g H_{eq} \nabla \cdot \mathbf{u} &= 0 \end{aligned} \qquad (LTE)$$