

12.086/12.586 Problem Set 2

Scale-free phenomena

Due: October 21

October 7, 2014

1. **White Collar Crime** One fantastically complex system is the accounting records of a large company. In addition to the multitude of mechanisms that influence accounts, people also lie. Is there any way one can identify “normal” accounts just by looking at them? One tool that is used to find evidence of “certain irregularities in accounting practices” is the probability distribution for the first digit of randomly chosen number. One may guess that the first digit of a randomly chosen stock price would be as likely to be 3 as to be 9; however empirically 1 occurs about 30% of the time while 9 occurs only about 5% of the time. There is an empirical p.d.f that describes this. To find cooked books, auditors compare the observed frequency of first digits from accounting records to the empirical p.d.f.. In fact, the first digit of numbers that describe many natural systems (e.g. river drainage areas, city populations, addresses, physical constants) all have what appears to be the same probability distribution. If one posits the existence of some universal probability distribution for the first digits of naturally occurring numbers, it is possible to calculate the observed probability distribution. The essential observation is that if there is a universal probability distribution function $P(x)$ for the first digit, then the shape of the function cannot change by rescaling the data (i.e. assume the probability that Microsoft’s stock is selling for something that starts with a 1 does not depend on whether we are trading in dollars or yen). Thus, $P(\lambda x) = f(\lambda)P(x)$.
 - (a) Use this constraint along with the requirement that P always be normalized to show that $f = \lambda^{-1}$ and $P \propto x^{-1}$.
 - (b) What is the probability that the first digit is a d ? (Hint: find the probability that a number picked between 1 and 10 will be between d and $d + 1$.)
 - (c) Benford (1938) collected data from a variety of different collections of numbers (e.g. addresses, physical constants, death rates). Here is the observed pdf for the first digit of data points from 741 various measures of the cost of things. $P(d = 1) = 32.4$, $P(d = 2) = 18.8$, $P(d = 3) = 10.1$, $P(d = 4) = 10.1$, $P(d = 5) = 9.8$, $P(d = 6) = 5.5$, $P(d = 7) = 4.7$, $P(d = 8) = 5.5$, $P(d = 9) = 3.1$. Compare this result to theory.
 - (d) (Optional) Generalize this result to first digits in base b .

2. **Diffusion on an ice cube tray** In class we considered the case of diffusion on a comb. We found that an unbiased random walker could get stuck as it wandered back and forth on a tooth of the comb. We again consider a random walker that gets stuck in a trap. The idea is that we place a marble into an ice cube tray and shake it randomly. If we shake it gently the marble spends a very long time in each hole in the tray. The harder we shake the tray, the more frequent the marble gets kicked into a new hole. If we shake the tray very hard the marble jumps on every shake. In this final case, the marble does not even “see” the holes and simply jumps randomly across the tray, resulting in an unbiased random walk.

- (a) The example of a marble in an infinite ice cube tray is clearly contrived. Give an example of a real system you suspect might be similar. (Example: Thermal agitation effectively shakes a system. The free-energy of a system can often be visualized with many local minima, in which a system can get stuck.)
- (b) Exponential distributions are common in nature (e.g. the time between the decay of radioactive atoms). They also describe the probability that particle taken from a system will have some specific energy. For example, the average nitrogen molecule on Earth is moving at about 500 m sec^{-1} , however there are some moving at 50000 m sec^{-1} . The probability that we can find such a molecule is proportional to $e^{-E/k_B T}$, where E is the kinetic energy of an N_2 molecule moving at 50000 m sec^{-1} , T is temperature and K_b is the Boltzmann constant. This observation gives rise to Arrhenius’ Law which predicts the time τ a particle spends in a well of depth V is proportional to $e^{V/k_B T}$. How does the diffusion constant depend on the temperature of the particle (i.e. how hard we are shaking the tray)? How does the mean-square distance of the marble scale with time? Give a plot of the scaling exponent as a function of the temperature of the particle.
- (c) To make the system sub-diffusive we need to add randomness to topography as well. Suppose that the probability of landing in a well of depth V is e^{-V/V_0} , where V_0 is the average depth of a well. (If you like, you can interpret V_0 as the thermal energy of the topography just like $k_B T$ is the thermal energy of the particle). Given that τ is a function of V , use this probability distribution to show that the probability the marble is stuck in a well for a time τ is proportional to $\tau^{-(1+\mu)}$. What is μ ?
- (d) How does the mean-square distance of the marble scale with time? Give a plot of the scaling exponent as a function of the temperature of the particle (Hint: For what values of μ is $\langle \tau \rangle$ finite?).
- (e) (Optional) Can you think of a physical system in which the step sizes have a power law distribution as would be the case for a Lévy walk?
- (f) Recall that $\langle X^2 \rangle = \langle \ell^2 \rangle N$. Combine the prediction of $\langle \ell^2 \rangle$ from Lévy walks with this result to find the scaling of a $\langle X^2 \rangle$ in time for a continuous time random walk with power law step sizes.

3. **Particle aggregation** In this problem we study randomly aggregating particles. Specifically, we are concerned with the particle mass distribution. Consider a sim-

ple model where particles occupy a finite 1-D lattice. At each time step, each particle can randomly jump to the next lattice site with probability $p > 0$, or remain in its current position. After all the particles have had a chance to jump, if two particles occupy the same site, they are merged into one particle whose mass is the sum of masses of the combined particles. We say the system is in steady-state when the particle mass distribution does not change with time.

- (a) What is the steady-state particle mass distribution if the system has:
- i. Absorbing boundaries (particles that jump from the last lattice site are removed from the system)?
 - ii. Periodic boundaries? (particles can jump from the last lattice site to the first lattice position)?

To make the problem more interesting, we inject new particles into the system. At each time step, after particles have jumped and merged, empty lattice sites are populated with particles of unit mass. The addition of particles to the system changes the steady-state particle mass distribution, such that it follows the power-law

$$p(m) \propto m^{-\alpha}, \tag{1}$$

where m is a mass.

- (b) Simulate such a system of aggregating particles with injection and periodic boundaries until it reaches steady-state (10,000 time steps and a lattice of size 1000 should work). Start your simulation from a lattice that is uniformly occupied with particles of mass 1, and set the jump probability $p = 0.5$. Use your simulation to estimate the exponent of particle mass distribution α .
- (c) Provide a theoretical explanation for the scaling law (1) and your estimate of α . (Hint: consider any similarities with the Scheidegger model discussed in class.)

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