MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

1.731 Water Resource Systems

Lecture 4, General Optimization Concepts 2, Sept. 19, 2006

When is a **local** optimum also a **global** optimum?

A **local** maximum/ minimum is a **global** maximum/minimum over the feasible region $\mathcal{F}$ if:

1. The feasible region is **convex**
2. The objective function is **convex** (for a **maximum**) or **concave** (for a **minimum**)

If the objective function is **strictly** convex or concave, the optimum is **unique**.

We need to **define terms** to apply this criterion.

Vector functions and derivatives:
Use vector notation used to represent multiple functions of multiple variables:

$$y = g(x) \rightarrow g_i(x_j) = g_i(x_1, x_2, ..., x_n) \quad i = 1, ..., m$$

Selected derivatives of scalar and vector functions:

Gradient vector of scalar function $f(\underline{x})$: $\dfrac{\partial f(x_1, ..., x_n)}{\partial x_i}$

Hessian matrix of scalar function $f(\underline{x})$: $\dfrac{\partial f(x_1, ..., x_n)}{\partial x_i \partial x_j}$ (symmetric)

Jacobian matrix of vector function $\underline{g_i}(\underline{x})$ $\dfrac{\partial g_i(x_1, ..., x_n)}{\partial x_j}$

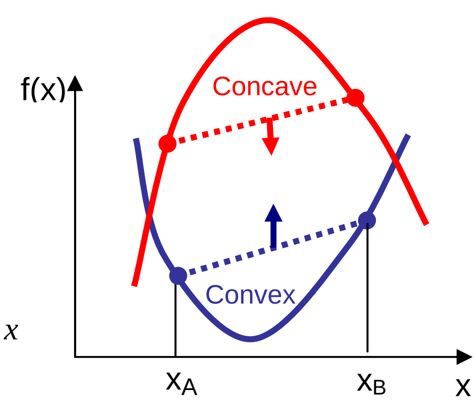
Convex/concave functions
Convexity of functions can be defined **geometrically** or in terms of **Hessian**:

***f(x)* is a convex function if:**

$$f[\alpha x_A + (1-\alpha)x_B] \leq f[\alpha x_A] + (1-\alpha) f[x_B]$$

Function lies **below** line connecting 2 points

$$H_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_i}$$ Hessian positive semi-definite $\forall\ x$



$f(x)$ is a concave function if:

$$f[\alpha x_A + (1-\alpha)x_B] \geq f[\alpha x_A] + (1-\alpha) f[x_B]$$

Function lies **above** line connecting 2 points

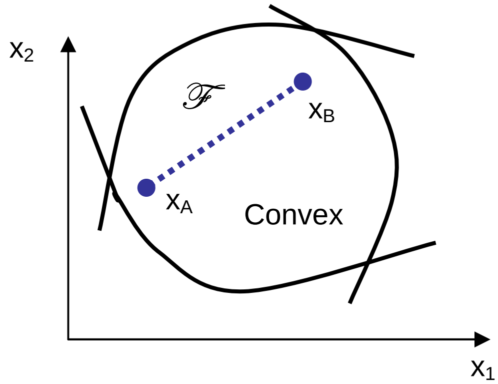
Linear functions are **both** convex and concave !

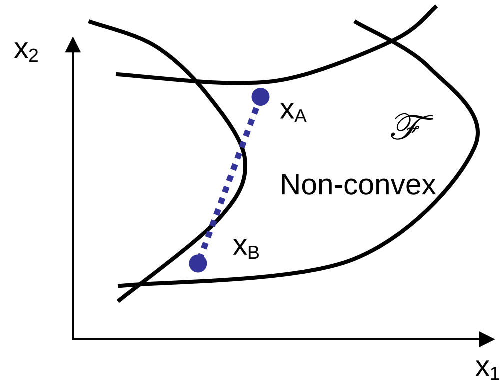
$$H_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_i} \quad \text{Hessian negative semi-definite } \forall\, x$$

Convex feasible region $\mathscr{F}$:

$\mathscr{F}$ is convex if line connecting any pair of points (x_A, x_B) lies completely inside region:

$$\alpha x_A + (1-\alpha)x_B \in \mathscr{F} \qquad \text{for all } (x_A, x_B) \text{ in } \mathscr{F} \quad \alpha \in [0,1]$$

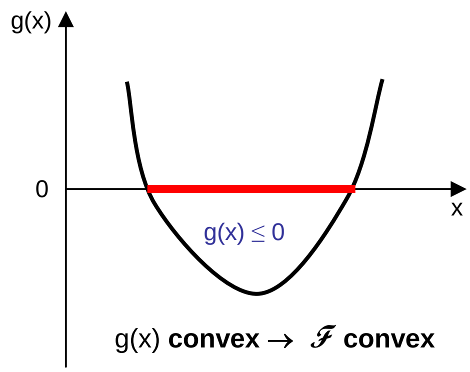




Convex feasible region may be constructed from m constraints that meet following requirements:

All $g_i(x)$ are **convex** when $g_i(x) \leq 0$
Or:
All $g_i(x)$ are **concave** when $g_i(x) \geq 0$

Feasible regions constructed from **linear** functions are **always convex**.



Summary:

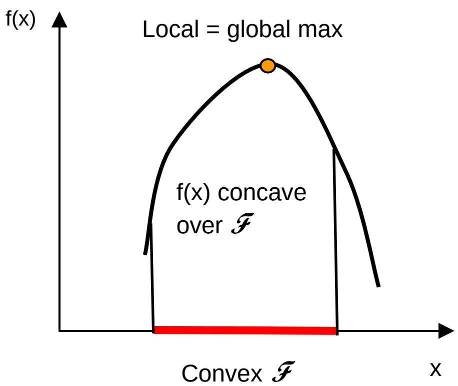
A **local** maximum/ minimum is a **global** maximum/minimum over the feasible region $\mathscr{F}$ if:

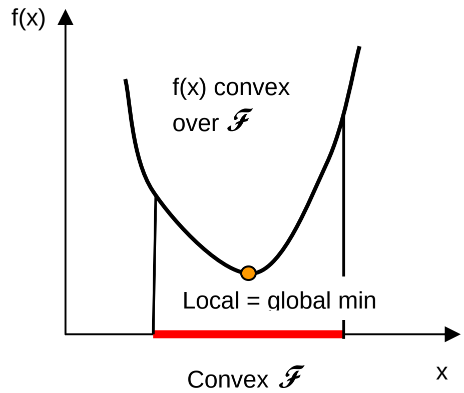
1. The feasible region is **convex**
2. The objective function is **convex** (for a **maximum**) or **concave** (for a **minimum**)

If the objective function is **strictly** convex or concave, the optimum is **unique**.

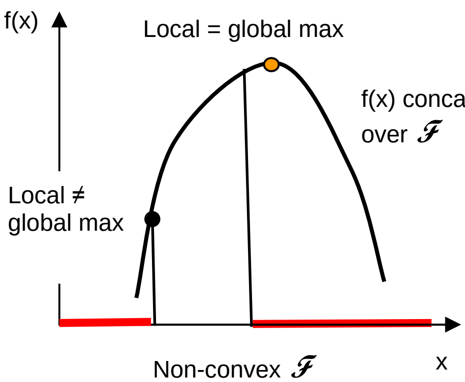
1D Examples:

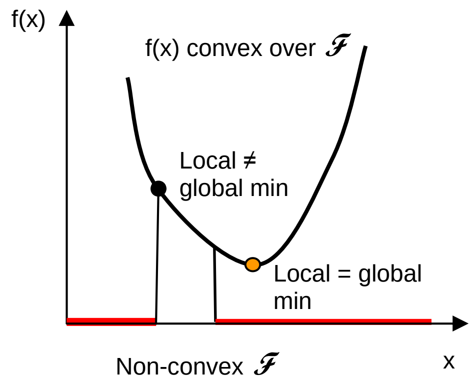
1. Objective is **convex/concave**, feasible region is **convex** $\rightarrow$ **local** maxima/minima **are global** maxima/minima.





2. Objective is **convex/concave**, feasible region is **not convex** $\rightarrow$ **local** maxima/minima **are not** necessarily **global** maxima/minima.





3. Objective is **not convex/concave**, feasible region is **convex** $\rightarrow$ **local** maxima/minima **are not** necessarily **global** maxima/minima.