Problem Set #4

Due: November 8, 2006

Problem 1

We have a single-server queueing system with infinite queue capacity. Arrivals of customers to the system occur in a Poisson manner at the rate of 36 per hour, that is $\frac{36}{60} = \frac{3}{5}$ per miniute. The service times, S, of customers are mutually independent and their duration is uniformly distributed between 1 and 2 minutes.

1. This is an M/G/1 system, and we can use the results derived in class for L_q and W_q. Recall that for a uniform distribution we have

$$E[S] = \frac{(a+b)}{2} \text{ and } \sigma^2 = \frac{(b-a)^2}{12}$$

In our case we have, $E[S] = 1.5$, $\sigma^2 = \frac{1}{12}$, $\rho = \frac{3}{2} \cdot \frac{3}{5} = \frac{9}{10}$. Also by definition $C_s = \frac{\sigma_s}{E[S]} = \frac{\sqrt{1/12}}{3/2} = \frac{2}{3\sqrt{12}}$, which gives us $C_s^2 = \frac{1}{27}$. Therefore

$$\begin{aligned} W_q &= \tfrac{1}{\mu} \cdot \tfrac{\rho}{(1-\rho)} \cdot \tfrac{(1+C_s^2)}{2} \\ &= \tfrac{3}{2} \cdot \tfrac{9/10}{1/10} \cdot \tfrac{(28/27)}{2} \\ &= 7 \end{aligned}$$

$$\begin{aligned} L_q &= \tfrac{\rho^2+\lambda^2\cdot\sigma_s^2}{2(1-\rho)} \\ &= \tfrac{(9/10)^2+(3/5)^2\cdot(1/12)}{2(1/10)} \\ &= \tfrac{21}{5} \end{aligned}$$

2. When the service times have negative exponential probability density, our M/G/1 queue is a M/M/1 queue. We can therefore either use the results for M/M/1 or M/G/1 queueing systems. We choose to contine to use the formulas from (a). Note that for a negative exponential distribution $\sigma_s = E[S] = \frac{3}{2}$ and therefore $C(S) = 1$, $\rho = \frac{9}{10}$ as before.

$$\begin{aligned} W_q &= \tfrac{1}{\mu} \cdot \tfrac{\rho}{(1-\rho)} \cdot \tfrac{(1+C_s^2)}{2} \\ &= \tfrac{1}{\mu} \cdot \tfrac{\rho}{(1-\rho)} \cdot 1 \\ &= \tfrac{3}{2} \cdot \tfrac{9/10}{1/10} \\ &= 13\tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} L_q &= \tfrac{\rho^2+\lambda^2\cdot\sigma_s^2}{2(1-\rho)} \\ &= \tfrac{(9/10)^2+(3/5)^2\cdot(3/2)^2}{2(1/10)} \\ &= \tfrac{81}{10} \end{aligned}$$

3. Now $\sigma_s = 0$ and we therefore $C_S = 0$.

$$\begin{aligned} W_q &= \frac{1}{\mu} \cdot \frac{\rho}{(1-\rho)} \cdot \frac{(1+C_s^2)}{2} \\ &= \frac{1}{\mu} \cdot \frac{\rho}{(1-\rho)} \cdot \frac{1}{2} \\ &= \frac{3}{2} \cdot \frac{9/10}{1/10} \cdot \frac{1}{2} \\ &= 6\frac{3}{4} \end{aligned}$$

$$\begin{aligned} L_q &= \frac{\rho^2 + \lambda^2 \cdot \sigma_s^2}{2(1-\rho)} \\ &= \frac{\rho^2}{2(1-\rho)} \\ &= \frac{(9/10)^2}{2(1/10)} \\ &= \frac{81}{20} \end{aligned}$$

4.

We use the bounds discussed in class: $B - \frac{1+r}{2l} \le W_q \le B$ with $B = \frac{l(s_X^2 + s_S^2)}{2(1-r)}$.

$s_X^2 = \frac{1}{l^2} = \frac{25}{9}$ and $s_S^2 = \frac{1}{12}$, therefore

$B = \frac{\frac{3}{5}(\frac{25}{9} + \frac{1}{12})}{2(1-0.9)} = 8.583$ and $\frac{1+r}{2l} = \frac{1+0.9}{2 \cdot \frac{3}{5}} = 1.583$

Thus, the upper and lower bounds are $7 \le W_q \le 8.583$.

By Little's Law, $4.2 \le L_q \le 5.150$.

5.

For exactly 4 customers to be in the system at the end next service, we need to have exactly three arrivals during the service of the next customer. Call this next service time t_1. Then the probability of three arrivals given t_1 is given by:

$$\frac{(\frac{3 \cdot t_1}{5})^3 \cdot e^{-3/5 \cdot t_1}}{3!}$$

We now need to evaluate this expression for all possible service times. The service times are Uniformly distributed between 1 and 2 and therefore $f_S(t) = 1 \quad 1 \le s \le 2$. Therefore:

$$\begin{aligned} &P(\text{4 customers in system at next epoch}) = \\ &P(\text{3 arrivals during next service}) = \\ &\int_1^2 \frac{(\frac{3 \cdot t_1}{5})^3 . e^{-3/5 \cdot t_1}}{3!} \cdot 1\, dt = \\ &\int_1^2 \frac{(\frac{3 \cdot t_1}{5})^3 . e^{-3/5 \cdot t_1}}{3!} dt \end{aligned}$$

6.

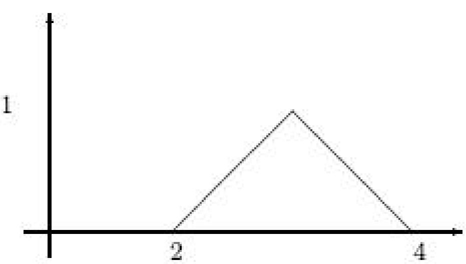
For there to be exactly four customers in the system after the next completion of service we need exactly four arrivals during the next service. Our expression therefore becomes:

$$
\begin{aligned}
&P(4 \text{ customers in system at next epoch}) = \\
&P(4 \text{ arrivals during next service}) = \\
&\int_1^2 \frac{(\frac{3 \cdot t_1}{5})^4 \cdot e^{-3/5 \cdot t_1}}{4!}\, dt
\end{aligned}
$$

7.

Now there are two customers in the system. The event that after their service there are exactly four customers in the system is the event that we have four arraivals during the two service times.

We first need to find the distribution of the two service times. Recall fram Problem set one, that the convolution of two uniforms has a triangilar shape density. Call $S_{tot} = S_1 + S_2$ then the density for S_{tot} is shown in the figure below:



$$
f_{S_{tot}}(t) = \begin{cases} t-2 & \text{for } 2 \le t \le 3 \\ -t+4 & \text{for } 3 \le t \le 4 \\ 0 & (otherwise) \end{cases}
$$

We are now ready to write our expression for the probability,

$$
\begin{aligned}
&P(4 \text{ customers in system at second epoch}) = \\
&P(4 \text{ arrivals during the next two services}) = \\
&\int_T \frac{(\frac{3 \cdot t}{5})^4 \cdot e^{-3/2 \cdot t}}{3!} \cdot f_{S_{tot}}(t)\, dt = \\
&\int_2^3 \frac{(\frac{3 \cdot t_1}{5})^4 \cdot e^{-3/5 \cdot t_1}}{4!} \cdot (t-2)dt + \int_3^4 \frac{(\frac{3 \cdot t_1}{5})^4 \cdot e^{-3/2 \cdot t_1}}{4!} \cdot (4-t)dt
\end{aligned}
$$

8.

Either George arrives during a busy period or during an idle period. Call the event that George arrives during a busy period A and the event that he arrives during a idle period A^c. We can then express George's expected waiting time using the total probability theorem:

$$E[W] = E[W|A] \cdot P(A) + E[W|A^c] \cdot P(A^c)$$

Now we know from a) that $P(A) = \frac{9}{10}$ and $P(A^c) = \frac{1}{10}$. If George arrives during a busy period we have to take random incidence into account. Using our random incidence formula for expected remainder (2.66 in Larson and Odoni) of an service period we have:

$$E[W|A] = \frac{\sigma_S^2 + E^2[S]}{2E[S]} = \frac{1/12 + (3/2)^2}{2(3/2)} = \frac{7}{9}$$

If George arrives in an idle period his expected waiting time is the expected time until the next arrival (which is $1/\lambda$) plus the expected service time, that is

$$E[W|A^c] = \frac{1}{\lambda} + E[S] = \frac{5}{3} + \frac{3}{2} = \frac{19}{6}$$

His total waiting time is therefore: $E[W] = E[W|A] \cdot P(A) + E[W|A^c] \cdot P(A^c) = \frac{7}{9}\frac{9}{10} + \frac{19}{6}\frac{1}{10} = 1\frac{1}{60}$

9.

Using results in ch 4.9 in Larson and Odoni we can calculate W_0:

$$W_0 = \sum_{i=1}^{2} \frac{\lambda_i \cdot E[S_i^2]}{2} = \frac{1}{2} \cdot \lambda E[S^2] = \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{7}{3} = \frac{7}{10}$$

Equation 4.107 (Larson & Odoni) gives us the expected wait in queue for differnt priority groups:

$$W_{qk} = \frac{W_0}{(1-a_{k-1})(1-a_k)} \quad \text{where } a_k = \sum_i^k \rho_i$$

$$\begin{aligned} W_{q1} &= \frac{\bar{W}_0}{(1-0)(1-\rho_1)} \\ &= \frac{\frac{7}{10}}{(1)(1-0.4 \cdot \frac{3}{5} \cdot \frac{3}{2})} \\ &= \frac{35}{32} \end{aligned}$$

$$\begin{aligned} \bar{W}_{q2} &= \frac{W_0}{(1-\rho_1)(1-\rho_1-\rho_2)} \\ &= \frac{W_0}{(1-\rho_1)(1-\rho)} \\ &= \frac{\frac{7}{10}}{(1-0.4 \cdot \frac{3}{5} \cdot \frac{3}{2}))(1-\frac{9}{10})} \\ &= \frac{175}{16} \end{aligned}$$

10.

We now have two classes of customers.

For Class A customers: $l_1 = 14.4$ per hour and $\begin{cases} E[S_1] = 1.2 \text{ min} \\ s_{S1}^2 = \dfrac{0.4^2}{12} \text{ min}^2 \end{cases}$. Thus

$$r_1 = \frac{14.4}{60/1.2} = 0.288.$$

For Class B customers: $l_2 = 21.6$ per hour and $\begin{cases} E[S_1] = 1.7 \text{ min} \\ s_{S1}^2 = \dfrac{0.6^2}{12} \text{ min}^2 \end{cases}$. Therefore,

$$r_2 = \frac{21.6}{60/1.7} = 0.612.$$

From Equation 4.107 of the L+O book, we have:

$W_{q1} = \dfrac{W_o}{1-0.288}$ and $W_{q2} = \dfrac{W_o}{(1-0.288)(1-0.288-0.612)}$

W_o is given by equation 4.102: $W_o = \dfrac{14.4\left(1.2^2 + \dfrac{0.04}{3}\right) + 21.6\left(1.7^2 + 0.03\right)}{2 \cdot 60}$.

Finally we have $W_q = 0.4 \cdot W_{q1} + 0.6 \cdot W_{q2}$ = 6.29 min.

In this new case, W_q is lower than in part 1 and part 9 because we are now giving the priority to the customers with the shortest expected time.
A corollary takes advantage of this phenomenon (p239 of the textbook):
To minimize the average waiting time for all users in the system, assign priorities according to the expected service times for each user class: the shorter the expected service time, the higher the priority of the class.

11.
This time, Class B customers have the priority. They have the greatest expected service time. Therefore, the overall W_q increases.

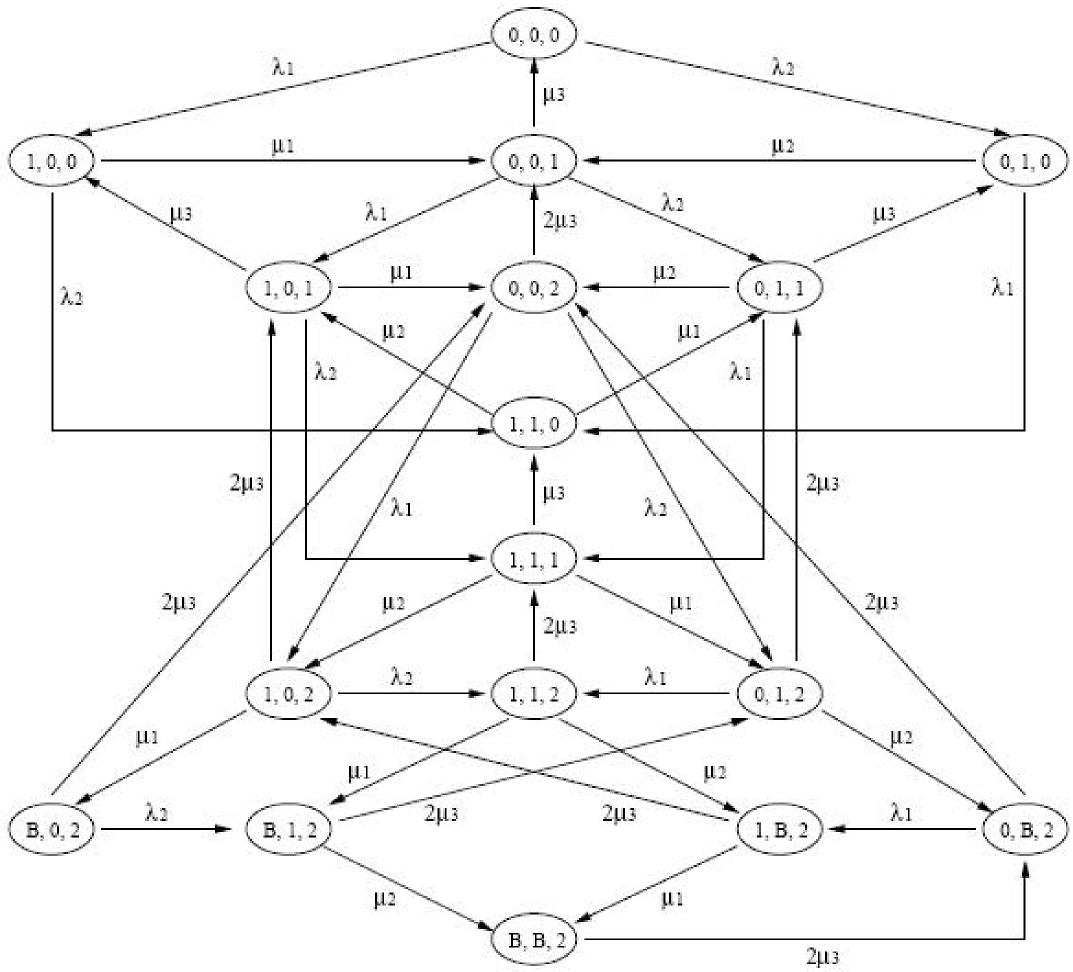
Numerical answer: W_q = 8.28

Problem 2

We define the states (i, j, k)
where:

- i describes System 1 and $i = 0, 1$ *or* b (for blocked)
- j describes System 2 and $j = 0, 1$ *or* b
- k describes Sytem 3 and $k = 0$ *or* 1

There are 13 states.
The corresponding state transition diagram is:



Problem 3

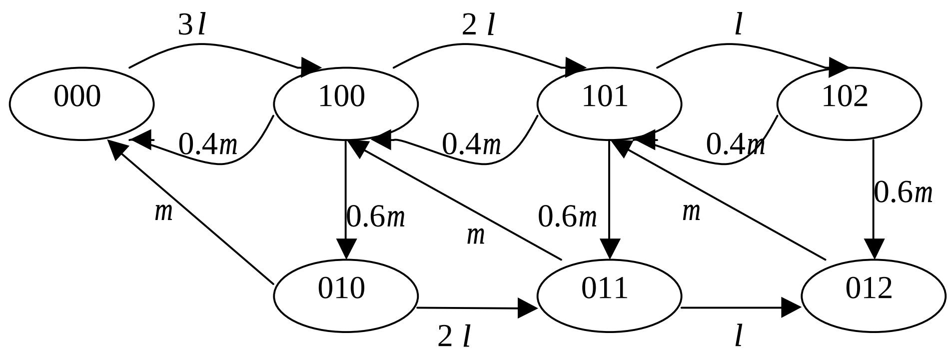
(a) We define the states (a, b, c)

Where:

a=number of planes in first stage of service

b=number of planes in second stage of service

c=number of planes in queue



(b) We need to compute the steady state probabilities. These can be found by solving the following set of equations:

$$2P(1,0,1) + 0.8P(1,1,0) = P(0,0,0)$$

$$P(0,0,0) + 0.8p(2,1,0) + 2P(2,0,1) = 1.2p(1,1,0) + 0.6666p(1,1,0) + .8p(1,1,0)$$

$$.66666P(1,1,0) + .8P(3,1,0) + 2P(3,0,1) = 1.2P(2,1,0) + .333333P(2,1,0)$$

$$.33333P(2,1,0) = .8P(3,1,0) + 1.2P(3,1,0)$$

$$1.2P(1,1,0) = 2P(1,0,1) + .66666P(1,0,1)$$

$$.6666P(1,0,1) + 1.2P(2,1,0) = 2P(2,0,1) + .33333P(2,0,1)$$

$$.33333P(2,0,1) + 1.2P(3,1,0) = 2P(3,0,1)$$

$$P(0,0,0) + P(1,1,0) + P(2,1,0) + P(3,1,0) + P(1,0,1) + P(2,0,1) + P(3,0,1) = 1$$

Solving these we get:

$$P(0,0,0) = .4224$$

$$P(1,1,0) = .2485$$

$$P(2,1,0) = .0964$$

$$P(3,1,0) = .0161$$

$$P(1,0,1) = .1118$$

$$P(2,0,1) = .0815$$

$$P(3,0,1) = .0232$$

Thus the expected value for the number in the system is:

$$(.2485 + .1118) + 2(.0964 + .0815) + 3(.0161 + .0232) = 0.834$$

Problem 4

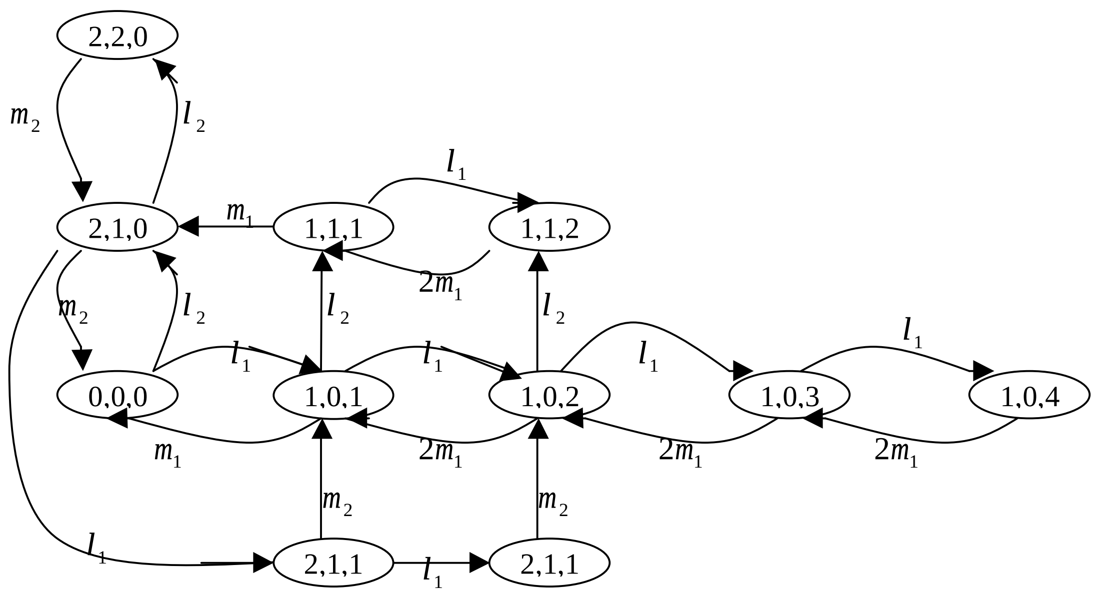
(a) A state is defined by (a,b,c) where:

a = the type of the customer(s) (0, 1, or 2) currently in service (you cannot have one server occupied by a Type 1 customer and the other server occupied by a Type 2 customer at the same time),

b = the number of Type 2 customers (0, 1, or 2) in the system,

c = the number of Type 1 customers (0, 1, 2, 3 or 4) in the system.

The corresponding state transition diagram for the system is:



(b) The event of interest can happen only by having a set of three consecutive transitions from state (1,0,4) to (1,0,3) to (1,0,2) to (1,1,2). We have:

$$P = \frac{2m_1}{2m_1 + l_1} \cdot \frac{l_2}{l_1 + l_2 + 2m_1}$$

Problem 5

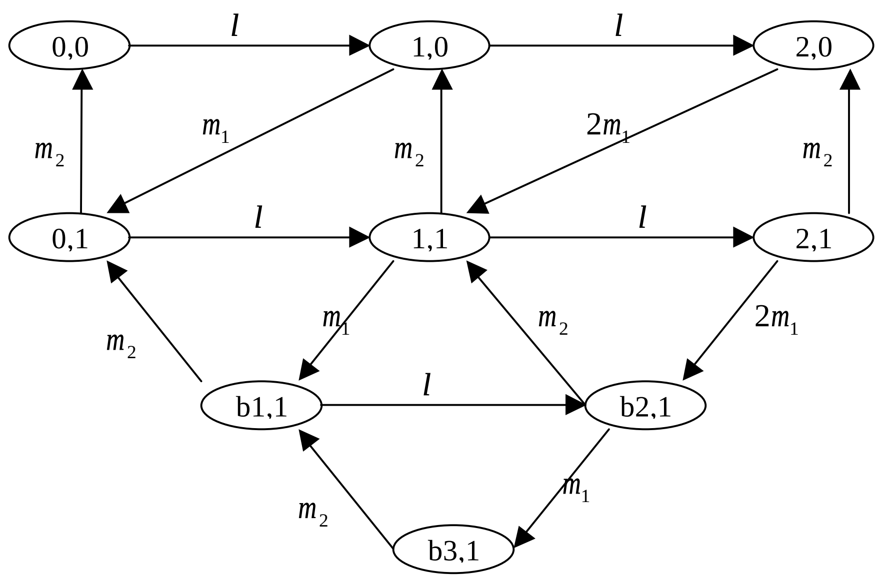
(c) A state is defined by (a,b) where a describes the first facility and b the second one.

a = 0 if there are no customers
1 if there is one customer
2 if there are 2 customers

b1 if there is one customer and he/she is blocked
b2 if there are two customers and one is blocked
b3 if there are two customers and both are blocked.

b = 0 if there are no customers
1 if there is one customer.

The corresponding state transition diagram for the system is:



(b) Clearly λ =(# in system)P(n in system)

$\lambda = 1*$P(1 in system)+2*P(2 in system)+3*P(3 in system)

λ =(P(1,0)+P(0,1))+2*(P(2,0)+P(1,1)+P(b1,1))+3*(P(2,1)+P(b2,1)+P(b3,1))