# LECTURE # 11

## 1.060 ENGINEERING MECHANICS II

### THE MOMENTUM PRINCIPLE

$$\frac{\partial}{\partial t}\int_{\forall_0} \rho \vec{q}\, d\forall = \int_{\forall_0} \rho \vec{g}\, d\forall + \sum \overrightarrow{MP} + \begin{Bmatrix}\text{Sum of all} \\ \text{other forces} \\ \text{acting \underline{on} fluid in } \forall_0\end{Bmatrix}$$

$\overrightarrow{MP}$ = Thrust (Momentum, $M$, & Pressure $P$) at inflow/outflow section of $\forall_0$, $A_{in}$ & $A_{out}$

If flow is well behaved at $A_{in}$ & $A_{out}$:
Straight parallel streamlines $\perp A$

then

$$\overrightarrow{MP} = \left[\int_{A_f} (\rho q_\perp^2 + p)\, dA\right](-\vec{n}_A) = \left[(K_m \rho V^2 + p_{CG}) A_f\right](-\vec{n}_{A_f})$$

$$K_m = \int_{A_f} q_\perp^2\, dA \big/ (V^2 A_f) \approx 1 \quad \text{(very well behaved)}$$

$V = Q/A_f$ ; $p_{CG}$ = pressure at Center of Gravity of $A_f$

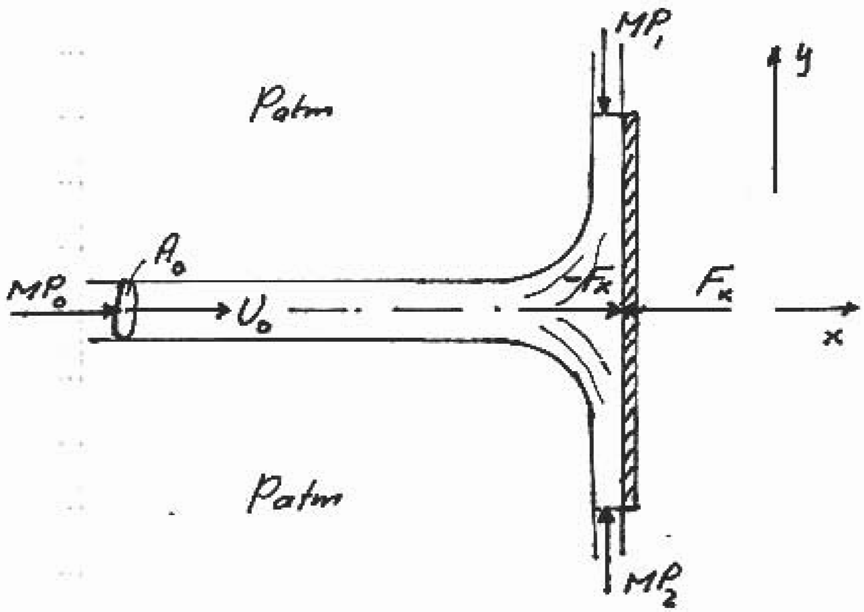
FOR STEADY FLOW : $\partial/\partial t = 0$

$$\int_{\forall_0} \rho \vec{g}\, d\forall + \sum \overrightarrow{MP} + \sum \overrightarrow{\text{all other forces}}_{\text{\underline{on} fluid in } \forall_0} = 0$$

$|\overrightarrow{MP}| = \rho V^2 A_f + p_{CG} A_f$ and it acts INWARDS TOWARDS $\forall_0$ in direction $\perp A_f$

$$\sum \overrightarrow{\text{Forces \underline{on} } \forall_0} = -\sum \overrightarrow{\text{Forces from } \forall_0}$$

EXAMPLE I: The Firehose




Gravity acts in $z$.

Plate wide enough to completely deflect the jet

Jet is freely "falling"

In x-direction

$$MP_{0x} + \underbrace{MP_{1x} + MP_{2x}}_{=0} + F_x = 0$$

$$F_x = \text{force } \underline{\text{on}} \text{ fluid from plate} = -MP_{0x} = -\underset{\substack{* \\ 0 \text{ since } p_0 = 0}}{p_0} A_0 - \rho U_0^2 A_0$$

$$\underline{-F_x} = \text{force from fluid on plate} = \underline{\rho U_0^2 A_0}$$

Firehose acts as a "pressure" force $(\rho U_0^2) A_0$

In y-direction

$$\underset{=0}{MP_{0y}} + MP_{1y} + MP_{2y} = 0 \Rightarrow MP_1 = MP_2$$

$p_1 = p_2 = 0$ since jets in atmosphere

$A_1 = A_2$ because of symmetry

$$(p_1 + \rho U_1^2) A_1 = (p_2 + \rho U_2^2) A_2 \Rightarrow \underline{U_1 = U_2}$$

But what is $U_1$ & $U_2$?

Bernoulli along surface streamlines from "0" to "1" and "0" to "2", with same z and $p_0 = p_1 = p_2 = 0$:

$$\frac{1}{2}\rho U_0^2 = \frac{1}{2}\rho U_1^2 \quad \text{and} \quad \frac{1}{2}\rho U_0^2 = \frac{1}{2}\rho U_2^2$$

$$\underline{U_0 = U_1 = U_2}$$

Note: If flow in xz-plane gravity enters the problem and we get from Bernoulli:

$$\rho g z_0 + \frac{1}{2}\rho U_0^2 = \rho g z_1 + \frac{1}{2}\rho U_1^2 = \rho g z_2 + \frac{1}{2}\rho U_2^2$$

or

$$U_1^2 = U_0^2 + 2g(z_0 - z_1) \; ; \; U_2^2 = U_0^2 + 2g(z_0 - z_2)$$

Thus,

$$U_0 \approx U_1 \approx U_2$$

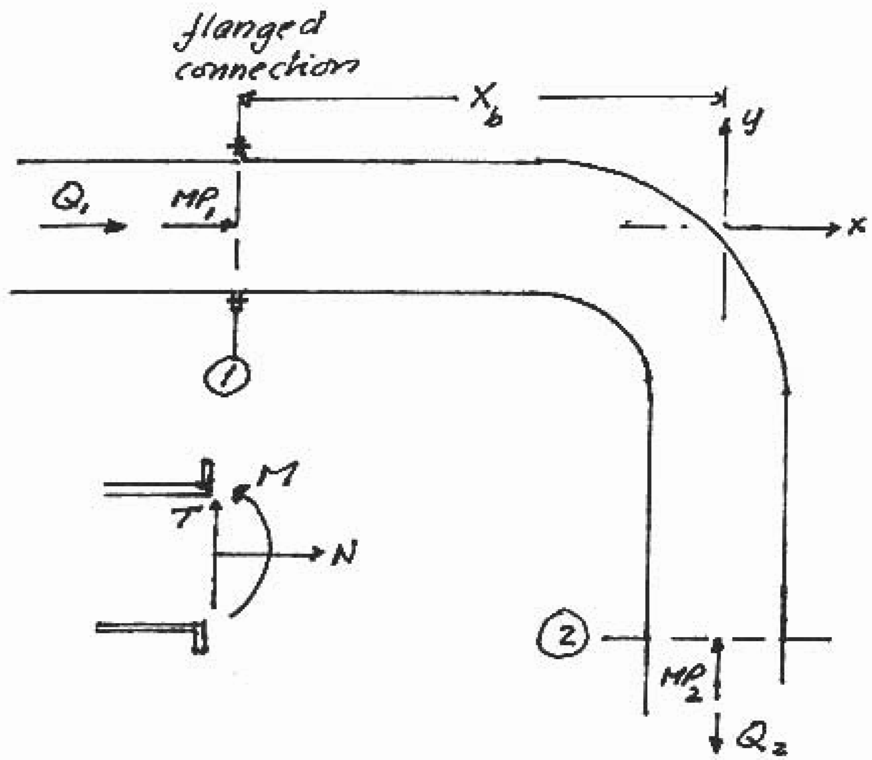
still hold if

$$\underline{\frac{U_0^2}{2g} = \text{velocity head} \gg \begin{Bmatrix} |z_0 - z_1| \\ |z_0 - z_2| \end{Bmatrix} = \text{elevation differenc}}$$

In problem where the characteristic velocity head, $V^2/2g$, is much larger than elevation differences, these may be neglected.

In momentum principle applications this translates into a neglect of gravity forces is acceptable when $V^2/2g \gg \Delta z$.

## EXAMPLE II : Flow around a bend




Gravity acts in z. (Neglect here)

Find forces & moment (in xy-plane) on flange at 1-1

Flow well behaved at 1-1 & 2-2 (far enough from bend)

Volume Conservation (always!)

$$Q_1 = Q_2 = Q \quad ; \quad A_1 = A_2 = A \quad \Rightarrow \quad V_1 = \frac{Q}{A_1} = V_2 = \frac{Q}{A_2} = V$$

Momentum in x-direction

$$MP_1 + F_x = 0 \Rightarrow F_x = -MP_1 = -(\rho V^2 + p_{CG,1})A$$

Force on Pipe = - Force on Fluid $= -F_x = \underline{(\rho V^2 + p_{CG,1})A = N}$

Momentum in y-direction

$$MP_2 + F_y = 0 \Rightarrow F_y = -MP_2 = -(\rho V^2 + p_{CG,2})A$$

Force on pipe $= -F_y = \underline{(\rho V^2 + p_{CG,2})A = T}$

Moment of forces around flanged connection

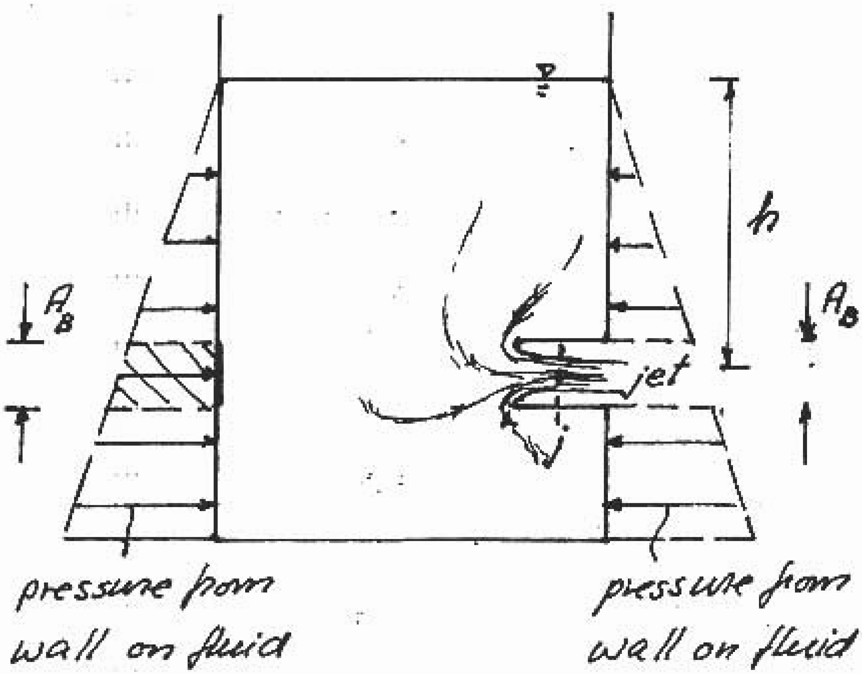
MP is acting as a total force - so just use it as such to get moments around CG of 1-1.

$MP_1$ has no moment arm $\Rightarrow$ $MP_1$ has no contribution

$MP_2$ has moment arm $= X_b$, so

$$\underline{M} = MP_2 \cdot X_b = \underline{(\rho V^2 + p_{CG,2}) A X_b} \quad \text{(counter clockwise)}$$

## EXAMPLE III : Borda's Mouthpiece




$A_B$ = area of Pipe outlet

$A_j$ = area of jet "after turning the corner" = Area at vena contracta.

<u>Bernoulli</u>

$$\rho g h = \tfrac{1}{2}\rho V_j^2 \Rightarrow \underline{V_j^2 = 2gh}$$

<u>Momentum</u>

Unbalanced pressure force from wall area $A_B$ opposite Borda's mouthpiece $= \rho g h A_B$ = Momentum force ($p_j = 0$ since jet is free) at vena contracta $= \rho V_j^2 A_j$. Or $\underline{V_j^2 = gh(A_B/A_j)}$

<u>Bernoulli & Momentum</u>

Same $V_j$ requires that

$$\underline{A_j / A_B = \text{Contraction Coefficient} = 0.5}$$

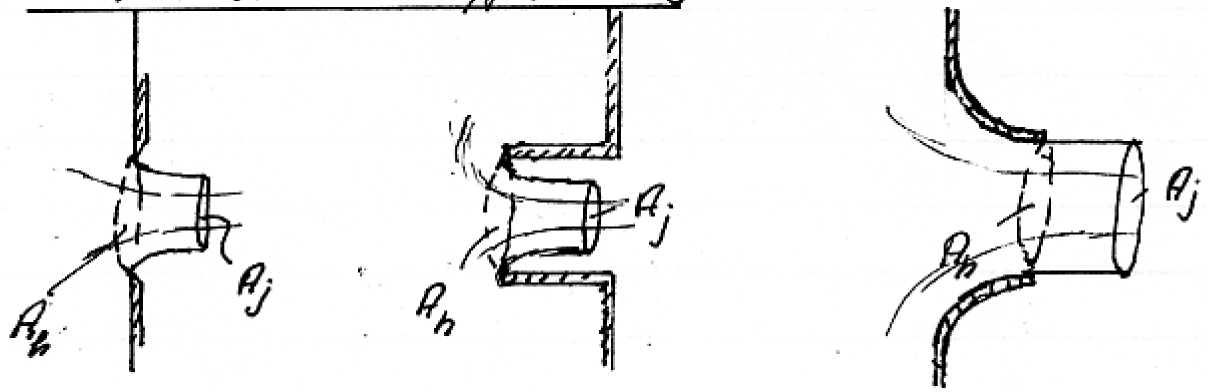
for Borda's Mouthpiece

<u>Reason why this works here</u>

Basic assumption is hydrostatic pressure along container walls everywhere, i.e. small velocities at walls. This holds here since $V \approx 0$ at the corner where pipe penetrates the container walls. Not so if orifice is in wall itself since $V \neq 0$ near hole.

# GENERAL OUTFLOW CONDITIONS

## Contraction Coefficients

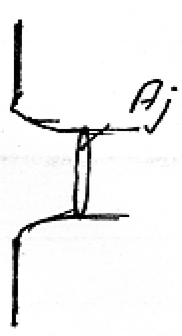



$C_v = A_j / A_h \cong 0.6$ $\quad C_v = 0.5$ $\quad C_v = 1.0$

Note: Contraction Coefficient refers to the ratio of AREAS of jet at vena contracta and orifice, not the diameter ratio (if circular orifice). For a 2-D orifice, i.e. a slot, Area = height · length and $C_v = (h_j \cdot \ell_j)/(h_h \ell_h) \cong h_j / h_h$ since $\ell_j \approx \ell_h$ if $\ell_h \gg h_h$. Looks like a "length" ratio, but it is really an "area per unit length" ratio.

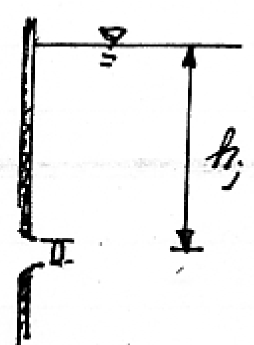
## Outflow Pressure Conditions

Free Outflow




$p_j = p_{atm} = 0$

Submerged Outflow




$p_j = \rho g h_j$

Generalized

Pressure at CG of jet at vena contracta = pressure in receiving body of fluid at that elevation.