

# Backprop for recurrent networks

# Steady state

- Reward is an explicit function of  $x$ .
- The steady state of a recurrent network.

$$x_i = f\left(\sum_j W_{ij}x_j + b_i\right)$$

$$\max_{W, b} R(x)$$

# Recurrent backpropagation

- Find steady state  $x = f(Wx + b)$
- Calculate slopes  $D = \text{diag}\{f'(Wx + b)\}$
- Solve for  $s$   $(D^{-1} - W^T)s = \frac{\partial R}{\partial x}$
- Weight update  $\Delta W = \eta s x^T$

# Sensitivity lemma

$$\frac{\partial R}{\partial W_{ij}} = \frac{\partial R}{\partial b_i} x_j$$

# Input as a function of output

- What input  $b$  is required to make  $x$  a steady state?

$$b_i = f^{-1}(x_i) - \sum_j W_{ij} x_j$$

- This is unique, even when output is not a unique function of the input!

# Jacobian matrix

$$b_i = f^{-1}(x_i) - \sum_j W_{ij} x_j$$

$$\frac{\partial b_i}{\partial x_j} = f^{-1'}(x_i) \delta_{ij} - W_{ij}$$

$$= (D^{-1} - W)_{ij}$$

# Chain rule

$$\begin{aligned}\frac{\partial R}{\partial x_j} &= \sum_i \frac{\partial R}{\partial b_i} \frac{\partial b_i}{\partial x_j} \\ &= \sum_i \frac{\partial R}{\partial b_i} (D^{-1} - W)_{ij}\end{aligned}$$

$$\frac{\partial R}{\partial x} = (D^{-1} - W^T) \frac{\partial R}{\partial b}$$

# Trajectories

- Initialize at  $x(0)$
- Iterate for  $T$  time steps

$$x_i(t) = f\left(\sum_j W_{ij}x_j(t-1) + b_i\right)$$

$$\max_{W, b} R(x(1), K, x(T))$$



# Unfold time into space

- Multilayer perceptron
  - Same number of neurons in each layer
  - Same weights and biases in each layer (weight-sharing)

$$x(0) \xrightarrow{W,b} x(1) \xrightarrow{W,b} \mathbf{L} \xrightarrow{W,b} x(T)$$

# Backpropagation through time

- Initial condition  $x(0)$

$$x(t) = f(Wx(t-1) + b(t))$$

- Compute  $R / \partial x(t)$
- Final condition  $s(T+1)=0$

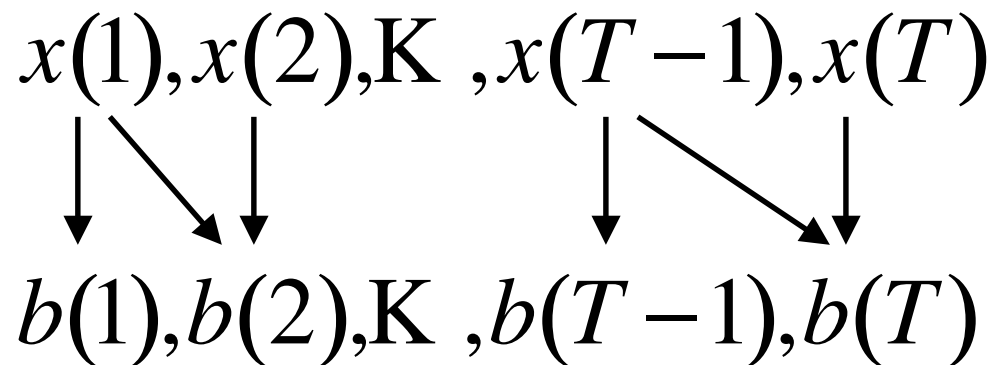
$$s(t) = D(t)W^T s(t+1) + D(t) \frac{\partial R}{\partial x(t)}$$

$$\Delta W = \eta \sum_t s(t)x(t-1)^T \quad \Delta b = \eta \sum_t s(t)$$

# Input as a function of output

$$x(t) = f(Wx(t-1) + b(t))$$

$$b(t) = f^{-1}(x(t)) - Wx(t-1)$$



# Jacobian matrix

$$b(t) = f^{-1}(x(t)) - Wx(t-1)$$

$$\frac{\partial b_i(t)}{\partial x_j(t')} = \delta_{tt'} (D^{-1}(t))_{ij} - W_{ij} \delta_{t-1,t'}$$

$$D(t) = \text{diag}\{f(Wx(t-1) + b(t))\}$$

# Chain rule

$$\begin{aligned}\frac{\partial R}{\partial x_j(t')} &= \sum_{i,t'} \frac{\partial R}{\partial b_i(t)} \frac{\partial b_i(t)}{\partial x_j(t')} \\ &= \sum_i s_i(t') (D^{-1}(t'))_{ij} - \sum_i s_i(t'+1) W_{ij}\end{aligned}$$

$$\frac{\partial R}{\partial x(t)} = D^{-1}(t) s(t) - W^T s(t+1)$$