16.61

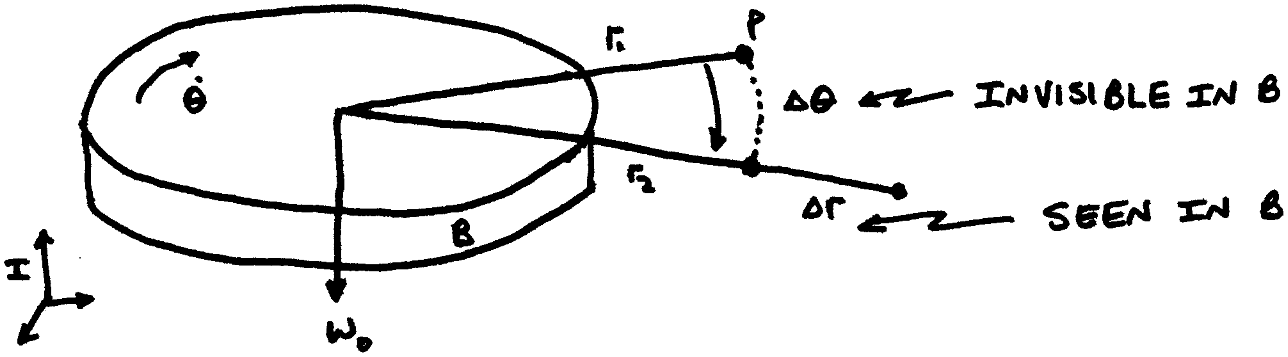
LECTURE #2

- CORIOLIS "DEMYSTIFIED"
- FRAMES
- EULER ANGLES
- ROTATIONS

GW 2-4, 2-5, 2-9

CORIOLIS ACCELERATION "DEMYSTIFIED"

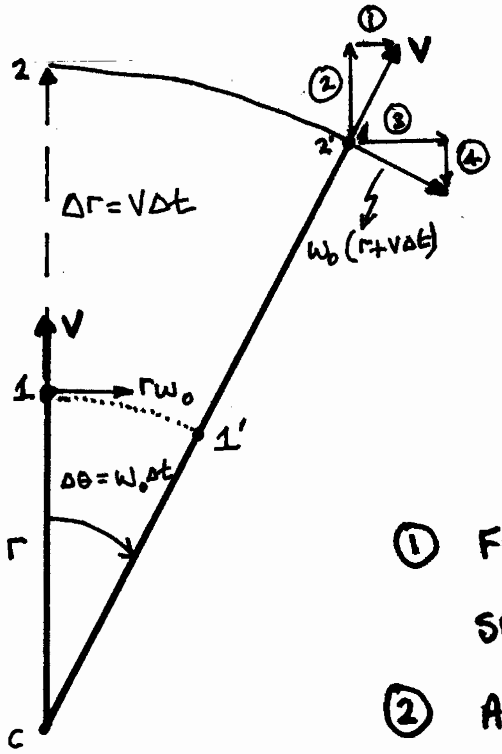
- CONSIDER CASE OF CONSTANT ROTATION, NO MOTION OF THE ORIGIN, AND CONSTANT RADIAL VELOCITY (AS SEEN IN THE ROTATING FRAME) $\ddot{\vec{r}}^I = \cancelto{0}{\ddot{\vec{r}}^B} + \cancelto{0}{\dot{\vec{\omega}}_0} \times \vec{r} + 2\vec{\omega}_0 \times \dot{\vec{r}}^B + \vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r})$



- SO THERE IS NO ACCELERATION OF PARTICLE P AS SEEN IN THE BODY FRAME.
- BUT THE ABSOLUTE ACCELERATION IS THE DIFFERENCE BETWEEN THE 2 VELOCITIES (ABSOLUTE) DIVIDED BY Δt. (AND THIS IS NON-ZERO).
  - ASSUME Δt SMALL, $\Delta\theta = \omega_0 \Delta t$
    $\sin \omega \Delta t \sim \omega_0 \Delta t$, $\cos \omega_0 \Delta t \sim 1$

SEE:
- WILLIAMS 93
- DEN HARTOG "MECHANICS" 302

- VIEW FROM ABOVE:



- NEED TO CALCULATE THIS DIFFERENCE IN 2 DIRECTIONS
- TANGENTIAL / RADIAL AT INITIAL POINT.

NOTE: CHANGES IN VELOCITY THAT ARE INVISIBLE IN B:

① FURTHER OUT IN RADIUS, SO TANGENTIAL SPEED MUST BE GREATER BY $\Delta r \dot{\theta}$

② AT NEW ANGLE $\Delta w_0 = \Delta\theta$, THERE IS A CHANGE IN THE DIRECTION OF THE MOTION RELATIVE TO THE BODY

$$C \rightarrow 1 \Rightarrow C \rightarrow 1'$$

- DIRECTION PARALLEL TO C12

$$\Delta V = (② + ④) - V$$

$$= (V\cos w_0 t - W_0(r + V\Delta t)\sin w_0 \Delta t) - V = -w_0^2 r \Delta t$$

$$\Rightarrow \frac{\Delta V}{\Delta t} = -w_0^2 r \quad \Rightarrow \quad \dot{V}_{RADIAL} = -w^2 r$$

CENTRIPETAL PART

- PERPENDICULAR TO C12

$$\Delta V = ① + ③ - r w_0$$

$$= V\sin W_0 t + w_0(r + \Delta t V)\cos w_0 t - r w_0$$

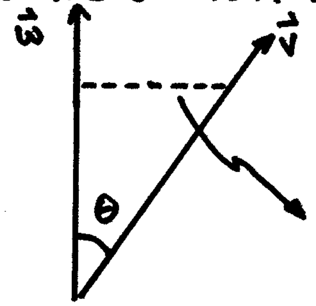
$$= V w_0^{\Delta} t + V w_0 \Delta t = 2 V w_0 \Delta t \quad \Rightarrow \quad \dot{V}_{TANG} = 2V w_0$$

- SO IT IS EXPLICIT THAT THE TANGENTIAL ACCELERATION IS THE CORIOLIS ACCELERATION (IN THIS CASE)
  - CONSISTS OF 2 CONTRIBUTIONS
    - CHANGE IN DIRECTION OF THE MOTION RELATIVE TO B (I.E. B FRAME ROTATES)
    - MOTION IN B CHANGES RADIUS, WHICH EFFECTS THE ROTATIONAL SPEED.

- ABSOLUTE ACCELERATION IS VECTOR SUM OF
  - RELATIVE ACCELERATION
  - CENTRIPETAL
  - CORIOLIS

  } CORRECTIONS MADE TO OBSERVATIONS BY OBSERVER IN A ROTATING FRAME.

- CORIOLIS TERM $2\vec{\omega}\times\dot{\vec{r}}^{B} - \vec{A}$



RECALL $|\vec{A}| = 2|\vec{\omega}||\dot{\vec{r}}^{B}|\sin\theta$

$|\vec{v}|\sin\theta \sim v_{\perp}$ VELOCITY COMPONENT **PERPENDICULAR** TO $\vec{\omega}$

- NOTE DIRECTION OF CORIOLIS ACCELERATION ALWAYS PERPENDICULAR TO BOTH $\vec{\omega}$ AND $\vec{v}$
  - "SMALL, BUT ALWAYS THERE"
  - "CHANGE IN DIRECTION"

CORRECTIONS BY A ROTATING OBSERVER

- WHEN ONE FRAME IS ROTATING WITH RESPECT TO ANOTHER, WE HAVE:

$$\vec{v}^I = \vec{v}^R + \vec{\omega} \times \vec{r}$$

$$\vec{a}^I = \vec{a}^R + 2(\vec{\omega} \times \vec{v}^R) + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$\vec{r}$: OFFSET FROM ORIGIN OF FRAMES

⇒ ASSUME ORIGINS OF TWO FRAMES ARE COINCIDENT.

- KEY POINT: NEWTON'S LAWS HOLD IN AN INERTIAL FRAME, SO CAN SIMPLY WRITE

$$\vec{F} = m\vec{a}^I$$

- BUT, IN THE ROTATING FRAME, THIS EXPANDS INTO

$$\vec{F} - 2m(\vec{\omega} \times \vec{v}_r) - m(\vec{\omega} \times (\vec{\omega} \times \vec{r})) = m\vec{a}^R$$

⇒ TO OBSERVER IN THE ROTATING SYSTEM, IT THUS APPEARS AS IF THE PARTICLE IS BEING ACTED UPON BY AN EFFECTIVE FORCE

$$\vec{F}_{EFF} = \vec{F} - 2m(\vec{\omega} \times \vec{v}_r) - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

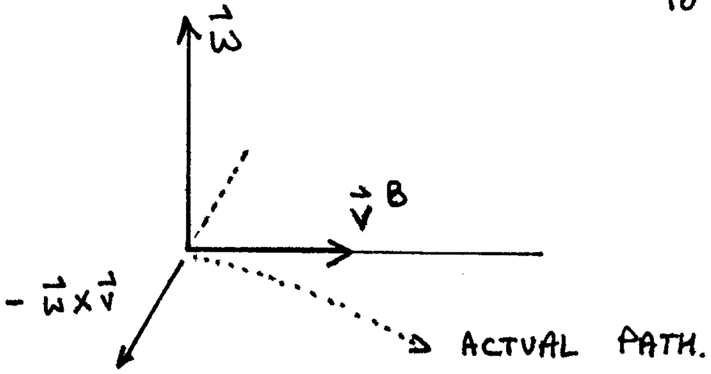
- THE ADDITIONAL "FICTITIOUS" FORCES ARE NEEDED TO "EXPLAIN" THE TRUE BEHAVIOUR OF THE PARTICLE, WHICH IS NOT SIMPLY JUST $\vec{F} = m\vec{a}^R$

$-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ CENTRIFUGAL FORCE

- ACTS RADIALLY

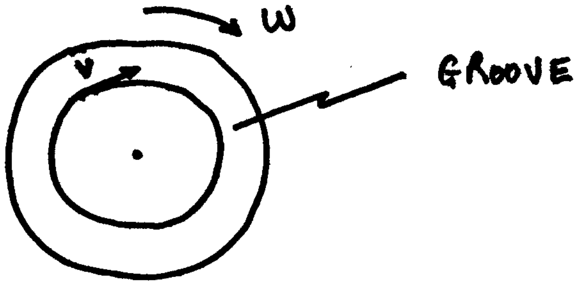
$-2m\vec{\omega} \times \vec{V}^R$ CORIOLIS FORCE

- ACTS ⊥ TO $\vec{\omega}, \vec{V}^R$ "TO THE RIGHT"



THREE CASES:

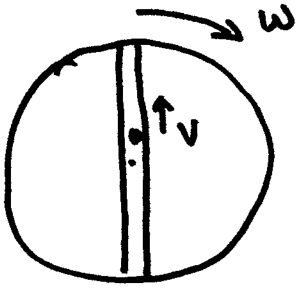
①



PARTICLE MOVES AROUND TRACK AT CONSTANT RELATIVE SPEED V

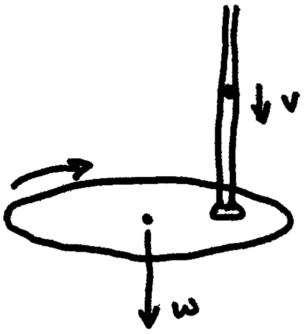
CORIOLIS ACCELERATION =

②



GROOVE NOW RADIAL

③



GROOVE NOW VERTICAL

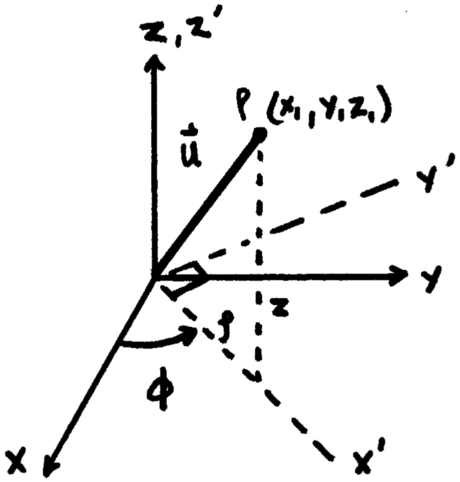
CORIOLIS ACCELERATION $2 \omega V_{\perp}$

$V_{\perp}$ - COMPONENT OF RELATIVE VELOCITY ⊥ TO AXIS OF ROTATION.

FRAMES OF REFERENCE

- WE HAVE THE BASIC EXPRESSION FOR THE INERTIAL (ABSOLUTE) ACCELERATION
  - NEED TO TRY IT OUT ON SOME FRAMES
  - CYLINDRICAL
  - SPHERICAL
  - GENERAL

CYLINDRICAL COORDINATES



HAVE $\vec{u}$ WITH COMPONENTS $u_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ IN THE FIXED x, y, z FRAME.

- SELECT SECOND FRAME x', y', z' SO THAT

$$u_2 = \begin{bmatrix} \rho \\ 0 \\ z \end{bmatrix} \longleftarrow$$

KEY IS THAT THIS TERM IS ZERO.

- THIS STEP IS ACCOMPLISHED BY ROTATING ABOUT THE $z\ (= z')$ AXIS BY ϕ FROM $x \Rightarrow x'$, $y \Rightarrow y'$

<u>ASIDE</u>: HOW CAN WE RELATE COMPONENTS WRT x, y, z AND x', y', z' ?

$\Rightarrow$ NEED ROTATION MATRICES.

NOTE: DENOTE BY $(\cdot)_1$ THE FACT THIS IS THE REPRESENTATION OF $\vec{u}$ WRT FRAME 1. AS OPPOSED TO $(\cdot)_2 \Rightarrow$ WRT FRAME 2.

- SO NOW WE HAVE A NEW SET OF COORDINATES TO DESCRIBE THE $\vec{u} \rightarrow \begin{bmatrix} \rho \\ \phi \\ z \end{bmatrix}$

- AS THE POINT P (TIP OF THE VECTOR $\vec{u}$) MOVES, THE FRAME WILL ROTATE TO MAINTAIN THE ALIGNMENT GIVEN PREVIOUSLY.

  ⇒ FRAME 2 WILL ROTATE ABOUT FRAME 1 WITH ANGULAR RATE $\vec{\omega} = \dot{\phi}\,\vec{K}$

  $\dot{\vec{\omega}} = \ddot{\phi}\,\vec{K}$

  $\Rightarrow \omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$

- THE ACCELERATION WRT THE INERTIAL FRAME IS THUS:

$$\ddot{\vec{u}}^I = \ddot{\vec{u}}^R + \dot{\vec{\omega}}^I \times \vec{u} + 2\vec{\omega} \times \dot{\vec{u}}^R + \vec{\omega} \times (\vec{\omega} \times \vec{u})$$

{USE MATRIX NOTATION}

$$\ddot{\vec{u}}^R \Rightarrow \begin{bmatrix} \ddot{\rho} \\ 0 \\ \ddot{z} \end{bmatrix} \longleftarrow \text{STILL ZERO BECAUSE OF THE ALIGNMENT}$$

$$\dot{\vec{\omega}}^I \times \vec{u} \Rightarrow \dot{\omega}_2^{\times} \begin{bmatrix} \rho \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 & -\ddot{\phi} & 0 \\ \ddot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \rho\ddot{\phi} \\ 0 \end{bmatrix}$$

* MAKE SURE ALL OF THE MATRIX COMPONENTS ARE WRITTEN WRT THE SAME FRAME *

$$2\vec{\omega}\times\dot{\vec{u}}^{R} = 2\,\omega_2^{\times}\begin{bmatrix}\dot{\rho}\\0\\\dot{z}\end{bmatrix} = 2\begin{bmatrix}0 & -\dot{\phi} & 0\\ \dot{\phi} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\rho}\\0\\\dot{z}\end{bmatrix} = \begin{bmatrix}0\\2\dot{\rho}\dot{\phi}\\0\end{bmatrix}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{u}) = \omega_2^{\times}\,\omega_2^{\times}\,u_2 = \begin{bmatrix}0 & -\dot{\phi} & 0\\ \dot{\phi} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & -\dot{\phi} & 0\\ \dot{\phi} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\rho\\0\\z\end{bmatrix}$$

$$= \begin{bmatrix}-\dot{\phi}^2 & 0 & 0\\ 0 & -\dot{\phi}^2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\rho\\0\\z\end{bmatrix} = \begin{bmatrix}-\rho\dot{\phi}^2\\0\\0\end{bmatrix}$$

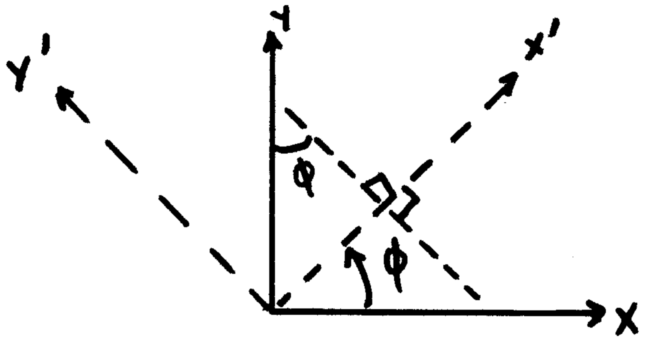
$$\therefore\quad \ddot{u}_2^{I} = \begin{bmatrix}\ddot{\rho}-\rho\dot{\phi}^2\\ \rho\ddot{\phi}+2\dot{\rho}\dot{\phi}\\ \ddot{z}\end{bmatrix}\begin{array}{l}\leftarrow \text{RADIAL COMPONENT}\\ \leftarrow \phi \text{ COMPONENT}\\ \leftarrow z \text{ COMPONENT}\end{array}$$

- TO MAP THE ACCELERATION INTO THE ORIGIONAL X,Y,Z FRAME, NEED TO USE THE ROTATION MATRIX

$$\ddot{u}_1^{I} = R_{12}\,\ddot{u}_2^{I}$$

ROTATION MATRIX

- LOOK DOWN Z-AXIS FROM ABOVE:



$$\left.\begin{aligned} X' &= X\cos\phi + Y\sin\phi \\ Y' &= -X\sin\phi + Y\cos\phi \\ Z' &= Z \end{aligned}\right\} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

A ROTATION MATRIX.
GIVEN VARIOUS SYMBOLS (R_{21})

- PROPERTIES OF ROTATION MATRICES

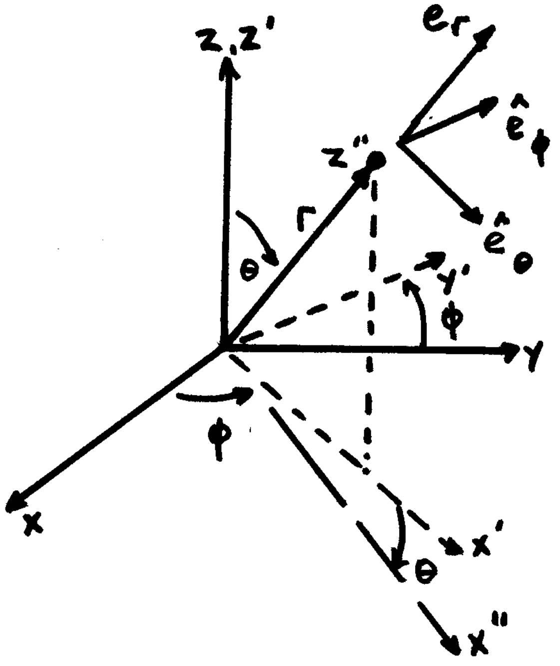
1) $\det(R_{ij}) = 1$

2) $R_{ij}^{-1} = R_{ji} = R_{ij}^{T} \longrightarrow R_{ij}^{T} R_{ij} = R_{ij} R_{ij}^{T} = I$

3) EITHER:

$$\begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ OR } \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & S\phi \\ 0 & -S\phi & C\phi \end{bmatrix} \text{ OR } \begin{bmatrix} C\phi & 0 & -S\phi \\ 0 & 1 & 0 \\ S\phi & 0 & C\phi \end{bmatrix}$$

SPHERICAL COORDINATES



IN THIS CASE, PERFORM 2 ROTATIONS SO THAT IN THE NEW FRAME THE COMPONENTS OF $\vec{r}$ ARE $r_S = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$

- FIRST BY ϕ ABOUT z
- THEN BY θ ABOUT y'

① $x \to x'$, $y \to y'$, $z = z'$ WHEN ROTATE BY ϕ

② $x' \to x''$, $y' = y''$, $z \to z''$ WHEN ROTATE BY θ

$x'' \to \hat{e}_\theta$

$y'' \to \hat{e}_\phi$

$z'' \to \hat{e}_r$

z'' ALIGNED WITH $\vec{r}$

- ANGULAR RATES :

① $\dot{\phi}$ ABOUT z

② $\dot{\theta}$ ABOUT y'

$\vec{\omega}$ IS VECTOR SUM OF THESE

- NEED COMPONENTS IN SPHERICAL FRAME x'', y'', z''

① $\dot{\phi}$ ABOUT z

$$\Rightarrow \begin{bmatrix} -\dot{\phi}\sin\theta \\ 0 \\ \dot{\phi}\cos\theta \end{bmatrix} \begin{matrix} x'' \\ y'' \\ z'' \end{matrix}$$

② $\dot{\theta}$ ABOUT $y' = y''$

$$\therefore \omega_S = \begin{bmatrix} -\dot{\phi}\sin\theta \\ \dot{\theta} \\ \dot{\phi}\cos\theta \end{bmatrix}$$

- OK, ONCE WE HAVE THE FRAMES AND THE ANGULAR RATE, THE REST IS PURELY MECHANICAL!

$$\omega_S = \begin{bmatrix} -\dot\phi \,\mathrm{SIN}\theta \\ \dot\theta \\ \dot\phi \,\mathrm{COS}\theta \end{bmatrix} ; \quad \dot\omega_S^I = \begin{bmatrix} -\ddot\phi \,\mathrm{SIN}\theta - \dot\phi \,\mathrm{COS}\theta\, \dot\theta \\ \ddot\theta \\ +\ddot\phi \,\mathrm{COS}\theta - \dot\phi \,\mathrm{SIN}\theta \dot\theta \end{bmatrix}$$

$$r_S = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} ; \quad \dot r_S^S = \begin{bmatrix} 0 \\ 0 \\ \dot r \end{bmatrix} ; \quad \ddot r_S^S = \begin{bmatrix} 0 \\ 0 \\ \ddot r \end{bmatrix}$$

- SO, THE ACCELERATION WRT INERTIAL SPACE IS GIVEN BY:

$$\ddot{\vec r}^{\,I} = \ddot{\vec r}^{\,s} + \dot{\vec\omega}^I \times \vec r + 2\vec\omega \times \dot{\vec r}^{\,s} + \vec\omega \times (\vec\omega \times \vec r)$$

EVALUATE IN MATRIX FORM (IN SPHERICAL COORDINATES)

$$\ddot{\vec r}^{\,s} \Rightarrow \ddot r_S^s$$

$$\dot{\vec\omega}^I \times \vec r \Rightarrow (\dot\omega_s^I)^\times r_S = \begin{bmatrix} 0 & -(\ddot\phi c\theta - \dot\phi\dot\theta s\theta) & \ddot\theta \\ (\ddot\phi c\theta - \dot\phi\dot\theta s\theta) & 0 & \ddot\phi s\theta + \dot\phi\dot\theta c\theta \\ -\ddot\theta & -\ddot\phi s\theta - \dot\phi\dot\theta c\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$$

$$= \begin{bmatrix} r\ddot\theta \\ r\ddot\phi s\theta + r\dot\phi\dot\theta c\theta \\ 0 \end{bmatrix}$$

$$2\vec{\omega}\times\dot{\vec{r}}^{\,s} \Rightarrow 2\begin{bmatrix} 0 & -\dot\phi c\theta & \dot\theta \\ \dot\phi c\theta & 0 & \dot\phi s\theta \\ -\dot\theta & -\dot\phi s\theta & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot r \end{bmatrix} = \begin{bmatrix} 2\dot r\dot\theta \\ 2\dot r\dot\phi s\theta \\ 0 \end{bmatrix}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) \Rightarrow \begin{bmatrix} 0 & -\dot\phi c\theta & \dot\theta \\ \dot\phi c\theta & 0 & \dot\phi s\theta \\ -\dot\theta & -\dot\phi s\theta & 0 \end{bmatrix}\begin{bmatrix} r\dot\theta \\ r\dot\phi s\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -r\dot\phi^2 s\theta c\theta \\ r\dot\theta\dot\phi c\theta \\ -r\dot\theta^2 - r\dot\phi^2 s^2\theta \end{bmatrix}$$

- COMBINE TO GET:

$$\ddot{r}^{I}_{S} = \left[\begin{array}{c} r\ddot\theta + 2\dot r\dot\theta - r\dot\phi^2 s\theta c\theta \\ r\ddot\phi s\theta + 2r\dot\phi\dot\theta c\theta + 2\dot r\dot\phi s\theta \\ \ddot r - r\dot\theta^2 - r\dot\phi^2 s^2\theta \end{array}\right] \qquad \begin{array}{c} \theta \\ \phi \\ r \end{array}$$

- AGAIN, CAN ROTATE BACK (2 ROTATIONS NOW) TO THE ORIGINAL FRAME TO GET $\ddot{r}^{I}_{I}$

EULER ANGLES   GW 7-13

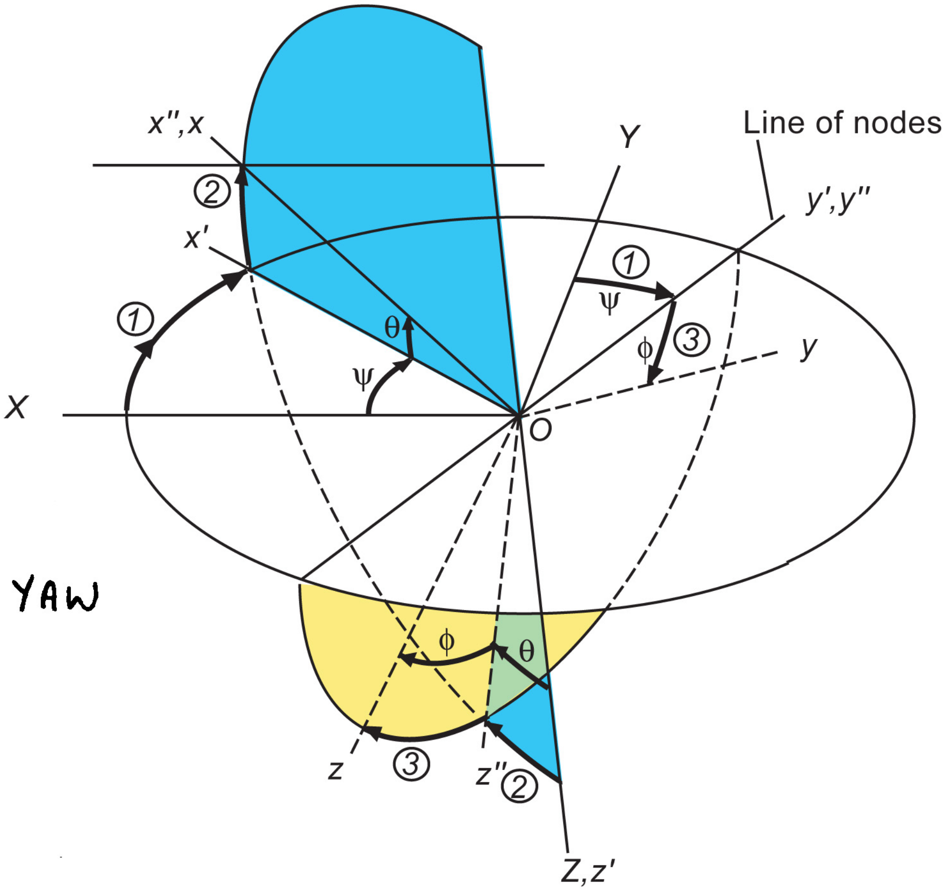
- FOR GENERAL APPLICATIONS IN 3D, WE OFTEN NEED TO PERFORM 3 SEPARATE ROTATIONS TO RELATE OUR INERTIAL FRAME TO OUR "BODY FRAME"
  - ⇒ ESPECIALLY TRUE FOR AIRCRAFT AND SPACECRAFT CASES.

- THERE ARE MANY WAYS TO DO THIS SET OF ROTATIONS (CHANGE ORDER OF ROTATIONS)
  - ALL WOULD BE ACCEPTABLE
  - SOME MORE COMMONLY USED THAN OTHERS.
- STANDARD :- START WITH BODY FRAME (xyz) ALIGNED WITH INERTIAL X,Y,Z
  - PERFORM 3 ROTATIONS TO RE-ORIENT BODY FRAME.

① ψ ABOUT Z
→ $x'y'z'$

② θ ABOUT y'
→ $x''y''z''$

③ ϕ ABOUT x''
→ XYZ



- **EULER ANGLES**

ψ ~ HEADING / YAW

θ ~ PITCH

ϕ ~ BANK

- CAN WRITE THESE ROTATIONS IN A CONVENIENT FORM :

$$\nearrow T_3(\psi)$$

① $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\nearrow T_2(\theta)$$

② $$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\nearrow T_1(\phi)$$

③ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \begin{bmatrix} x'' \\ y' \\ z'' \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = T_1(\phi) T_2(\theta) T_3(\psi) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi + s\phi s\theta s\psi & s\phi c\theta \\ s\phi s\psi + c\phi s\theta c\psi & -s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{bmatrix}$$

- NOTE THAT THE ORDER THAT THESE ROTATIONS ARE APPLIED MATTERS + WILL GREATLY CHANGE THE ANSWER - SO BE CAREFUL.

- TO GET THE ANGULAR VELOCITY IN THIS CASE, WE HAVE TO WORRY ABOUT THREE TERMS.

$$\left.\begin{array}{lll} ① & \dot{\psi} \text{ ABOUT } z \\ ② & \dot{\theta} \text{ ABOUT } y' \\ ③ & \dot{\phi} \text{ ABOUT } x'' \end{array}\right\} \text{COMBINE TO GET } \vec{\omega}$$

- NEED TO WRITE $\vec{\omega}$ IN TERMS OF ITS COMPONENTS IN THE FINAL FRAME (BODY FRAME)

  $\Rightarrow$ USE THE ROTATION MATRICES.

- EXAMPLE: $\dot{\psi}$ ABOUT $Z = z'$

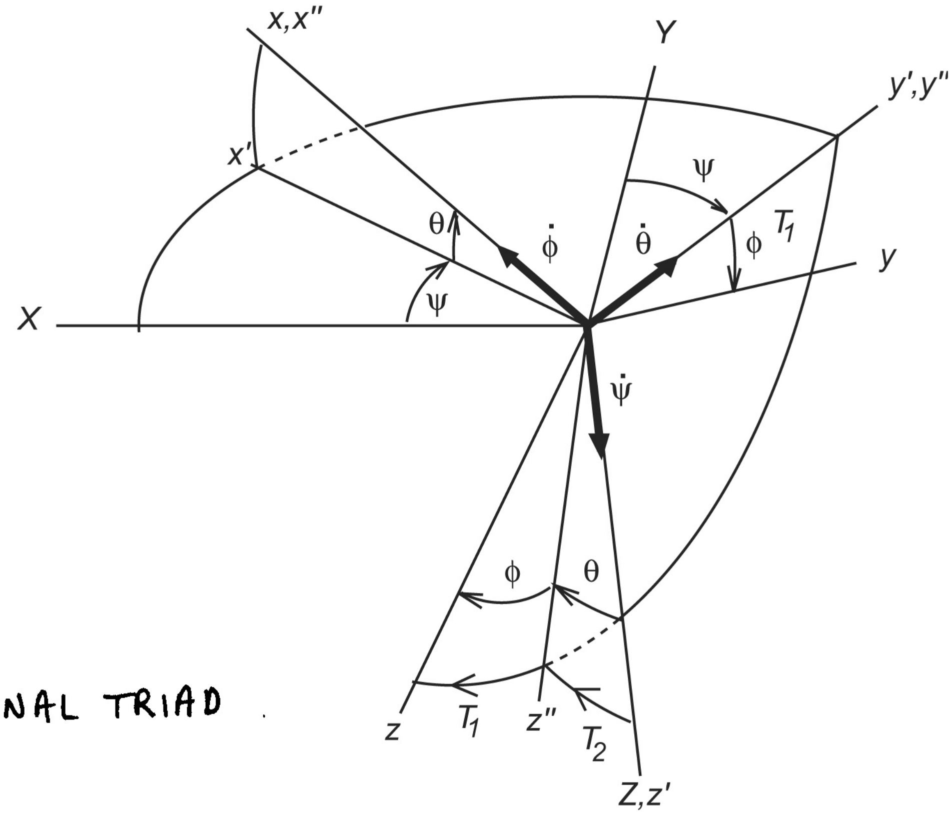
  - IN TERMS OF X, Y, Z FRAME ROTATION RATE HAS COMPONENTS $\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \leftarrow$ SAME IN x', y', z'

  - BUT TO TRANSFORM A VECTOR FROM $\begin{matrix} x' \\ y' \\ z' \end{matrix} \rightarrow \begin{matrix} x \\ y \\ z \end{matrix}$ WE WOULD USE $T_1(\phi)T_2(\theta)$

- SIMILARLY FOR $\dot{\theta}$ ABOUT $y' = y''$

  $\Rightarrow$ USE $T_1(\phi)$ ON $\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$

- **VISUALIZATION:**



- CAN WRITE

$$\vec{\omega} = \vec{\dot{\phi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}$$

- BUT $\vec{\dot{\psi}}, \vec{\dot{\theta}}, \vec{\dot{\phi}}$
DO NOT FORM A
MUTUALLY ORTHOGONAL TRIAD

⇒ NEED TO FORM THE
ORTHOGONAL PROJECTIONS ONTO THE
BODY FRAME XYZ

$$\omega_b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = T_1(\phi)T_2(\theta)\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + T_1(\phi)\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

- FINAL FORM:

$$\begin{aligned} \omega_x &= \dot{\phi} - \dot{\psi}\sin\theta \\ \omega_y &= \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ \omega_z &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi \end{aligned}$$

AND INVERSE

$$\begin{aligned} \dot{\phi} &= \omega_x + [\omega_y\sin\phi + \omega_z\cos\phi]\tan\theta \\ \dot{\theta} &= \omega_y\cos\phi - \omega_z\sin\phi \\ \dot{\psi} &= [\omega_y\sin\phi + \omega_z\cos\phi]\sec\theta \end{aligned}$$

WATCH FOR SINGULARITIES
AT $|\theta| = 90°$

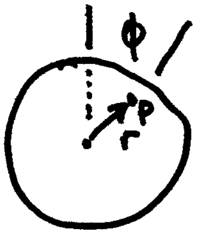
- IF WE LIMIT

$$0 \leq \psi \leq 2\pi$$
$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$
$$0 \leq \phi < 2\pi$$

THEN ANY POSSIBLE ORINTATION OF THE BODY CAN BE OBTAINED BY PERFORMING THE APPROPRIATE ROTATIONS IN THIS GIVEN ORDER.

- THESE ARE A PRETTY STANDARD SET OF EULER ANGLES
  - WE WILL USE FOR MOST OF THE A/C AND S/C WORK.

EXAMPLE: GW EX 2-4



PARTICLE P ON A DISK MOVES IN A SLOT SO THAT $r = \frac{a}{2}(1+\sin wt)$

- DISK ROTATES SO THAT $\phi(t) = \phi_0 \sin wt$

$$\Rightarrow \quad \dot{\phi} = \phi_0 w \cos wt$$
$$\ddot{\phi} = -\phi_0 w^2 \sin wt$$

- FIND ABSOLUTE ACCELERATION OF P?

- CYLINDRICAL COORDINATES

$$r_D = \begin{bmatrix} \frac{a}{2}(1+\sin wt) \\ 0 \\ 0 \end{bmatrix} ; \quad \dot{r}_D^D = \begin{bmatrix} \frac{aw}{2}\cos wt \\ 0 \\ 0 \end{bmatrix} ; \quad \ddot{r}_D^D = \begin{bmatrix} -\frac{aw^2}{2}\sin wt \\ 0 \\ 0 \end{bmatrix}$$

$$\ddot{r}_D^I = \begin{bmatrix} -\frac{aw^2}{2}\sin wt \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 & -\dot{\phi} & 0 \\ \dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{aw}{2}\cos wt \\ 0 \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\ddot{\phi} & 0 \\ \ddot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{a}{2}(1+\sin wt) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{\phi}^2 & 0 & 0 \\ 0 & -\dot{\phi}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{a}{2}(1+\sin wt) \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{aw^2}{2}\sin wt - \dot{\phi}^2 \frac{a}{2}(1+\sin wt) \\ 2\dot{\phi}\frac{aw}{2}\cos wt + \ddot{\phi}\frac{a}{2}(1+\sin wt) \\ 0 \end{bmatrix} \begin{matrix} \leftarrow \vec{r} \\ \leftarrow \vec{\phi} \\ \leftarrow \vec{z} \end{matrix}$$