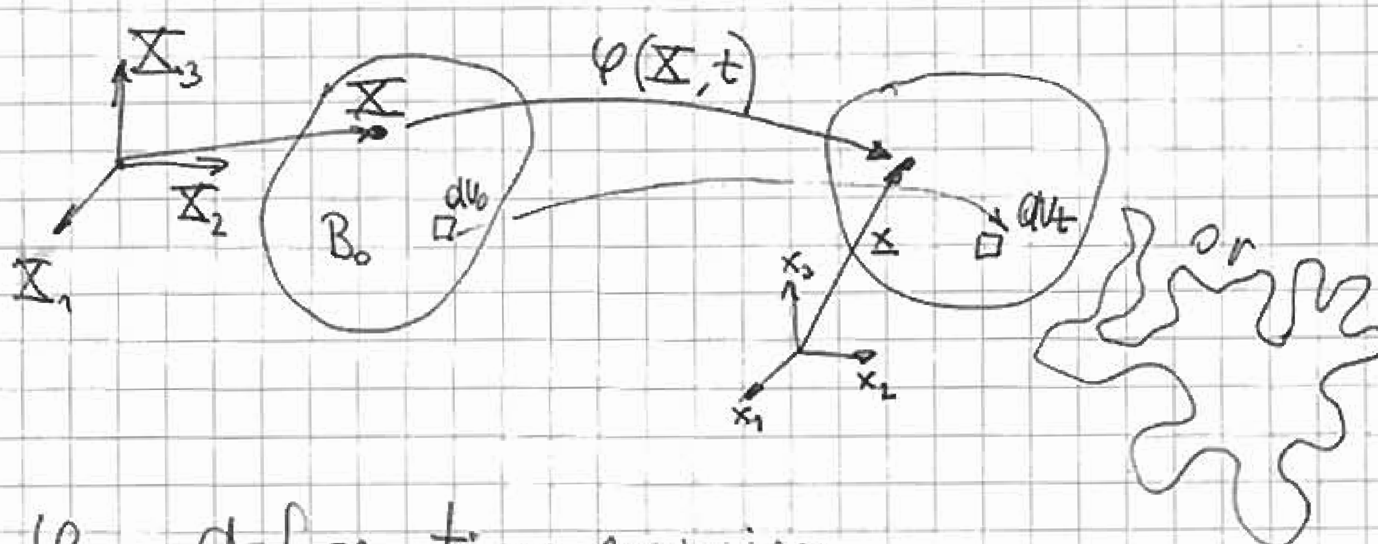
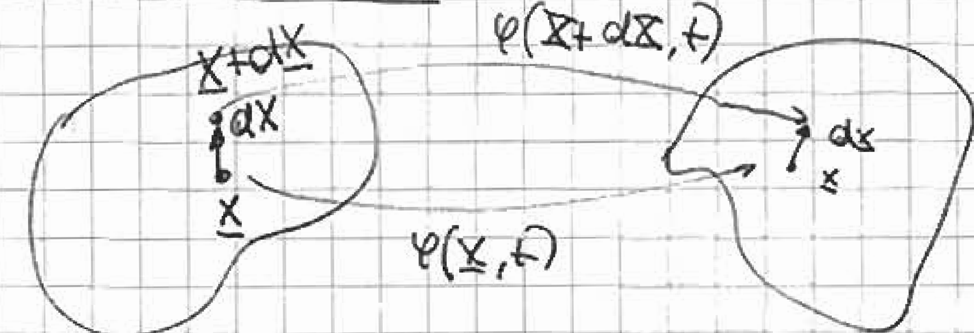
FINITE ELASTICITY



$\varphi \equiv$ deformation mapping

$$\boxed{x_i = \varphi_i(\underline{X}, t)}$$



$$d\underline{x} = \varphi(\underline{X} + d\underline{X}, t) - \varphi(\underline{X}, t) \sim$$

$$\underbrace{\frac{\partial \varphi_i}{\partial X_J}(\underline{X},t)}\, dX_J + \text{hot}$$

$F_{iJ}(\underline{X},t) \equiv$ def. gradient

Metric changes

$$ds^2 = dx_i\, dx_i = (F_{iI}\, dX_I)(F_{iJ}\, dX_J)$$

$$= \underbrace{F_{iI}\, F_{iJ}}\, dX_I\, dX_J$$

$C_{IJ} \equiv$ Cauchy-Green def. tensor

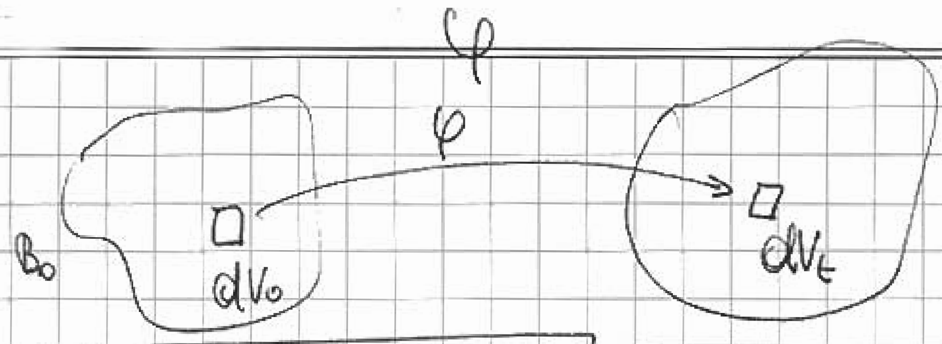
$$\boxed{C = F^T F}$$

$$dS^2 = dX_I\, dX_I$$

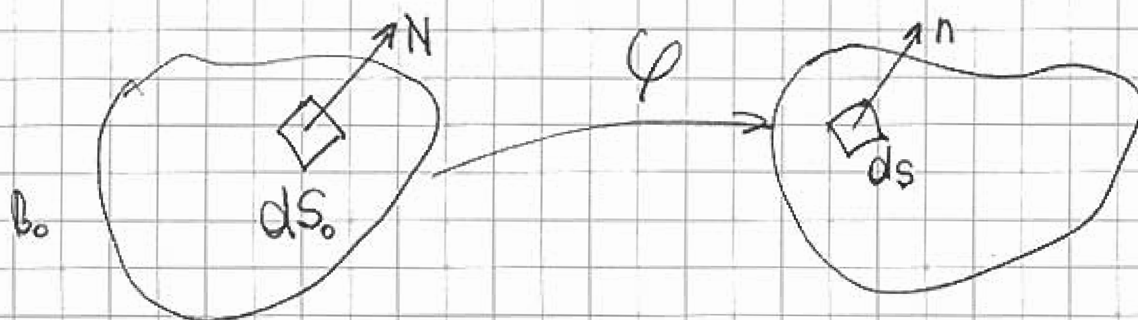
$$\frac{1}{2}(ds^2 - dS^2) = dX^T (C - I)\, dX \, \frac{1}{2}$$

$$= dX^T E\, dX$$

$$E_{IJ} = \frac{1}{2}(C_{IJ} - I_{IJ})$$

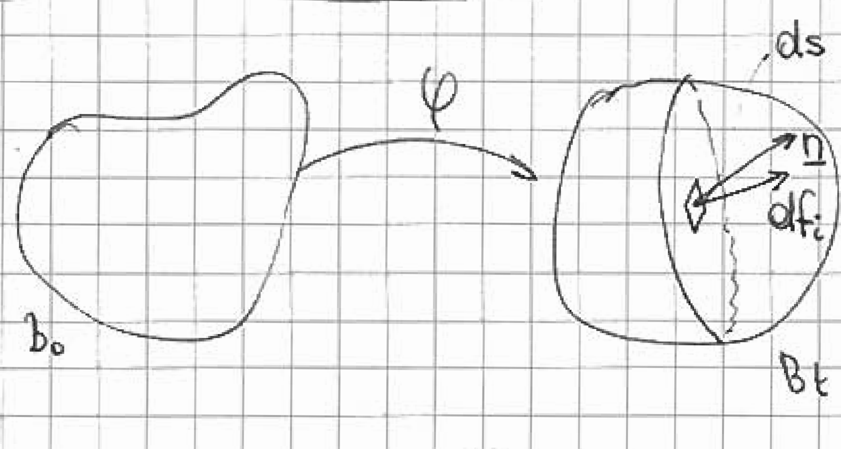


$$\boxed{\frac{dV}{dV_0} = \det(F) = J}$$ jacobian of def.



Piola transformation $\boxed{n_i\, ds = J\, N_I\, F^{-1}_{Ii}\, dS_0}$

State of stress



df_i = absolutely continuous wrt area

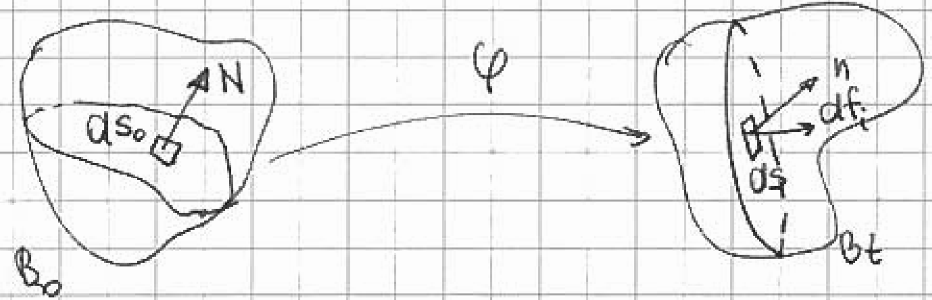
$$\boxed{df_i = t_i\, ds}$$

t_i: ~~tractions~~ spatial forces / unit deformed area

From Cauchy tetrahedron theorem:

$$\boxed{t_i = \sigma_{ij} n_j}$$

Cauchy stress tensor



Define material tractions $T_i = \dfrac{df_i}{dS_o}$

$$df_i = T_i \, dS_o = t_i \, ds = \sigma_{ij} n_j \, ds$$

$$= \sigma_{ij} J N_I F^{-1}_{Ij} \, dS_o \Longrightarrow$$

1st Piola Kirchhoff stress tensor

$$\boxed{T_i = P_{iI} N_I} \qquad \boxed{P_{iI} = J F^{-1}_{Ij} \sigma_{ij}}$$

$$\boxed{P = J\sigma F^{-T}}$$

Def: $df_I = F^{-1}_{Ij}\, df_j = S_{IJ} N_J\, dS_0$

2nd PK stress tensor $\boxed{S_{IJ} = J\, F^{-1}_{Ii}\, F^{-1}_{Jj}\, \sigma_{ij}}$ $\boxed{S = J F^{-1} \sigma F^{-T}}$

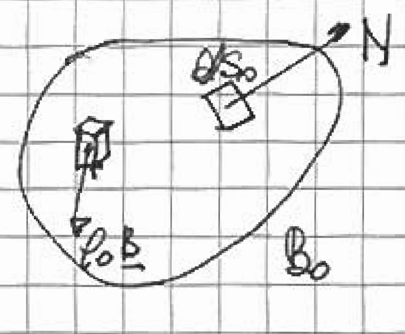
Kirchhoff stress $\boxed{\tau_{ij} = J\sigma_{ij}}$

Field equations

Linear momentum balance:

$$L_i = \int_{B_0} \rho_0 V_i \, dV_0$$

$$R_i = \int_{B_0} \rho_0 B_i \, dV_0 + \int_{S_0} \overbrace{P_{iI} N_I}^{T_i} \, dS_0$$



$$\frac{dL_i}{dt} = R_i \Longrightarrow \boxed{P_{iJ,J} + \rho_0 B_i = \rho_0 A_i}$$

$$\boxed{\nabla_0 \cdot P + \rho_0 B = \rho_0 A}$$

material gradient

$$V_i = \frac{\partial \varphi_i(X,t)}{\partial t} = V_i(X,t)$$

$$A_i = \frac{\partial V_i}{\partial t}(X,t)$$

$\mapsto$ defined over the reference config.
happening on the spatial velocity (i)

Angular Momentum balance

$$\boxed{\sigma = \sigma^T} \Longrightarrow$$

$$\boxed{S = S^T}$$

$$\boxed{\tau^T = \tau}$$

Energy balance

$$\boxed{\dot{u} = \underbrace{S_{IJ} \frac{1}{2} \dot{C}_{IJ}}_{} = P_{iJ} \dot{F}_{iJ} = \tau_{ij} d_{ij}}$$

deformation power / unit UNDEFORMED volume

d_{ij} = rate of def. tensor $= \frac{1}{2}(l_{ij} + l_{ji})$

$l_{ij} = v_{i,j} \equiv$ spatial velocity gradient

$$l = \dot{F} F^{-1}$$

Nonlinear elastic solid

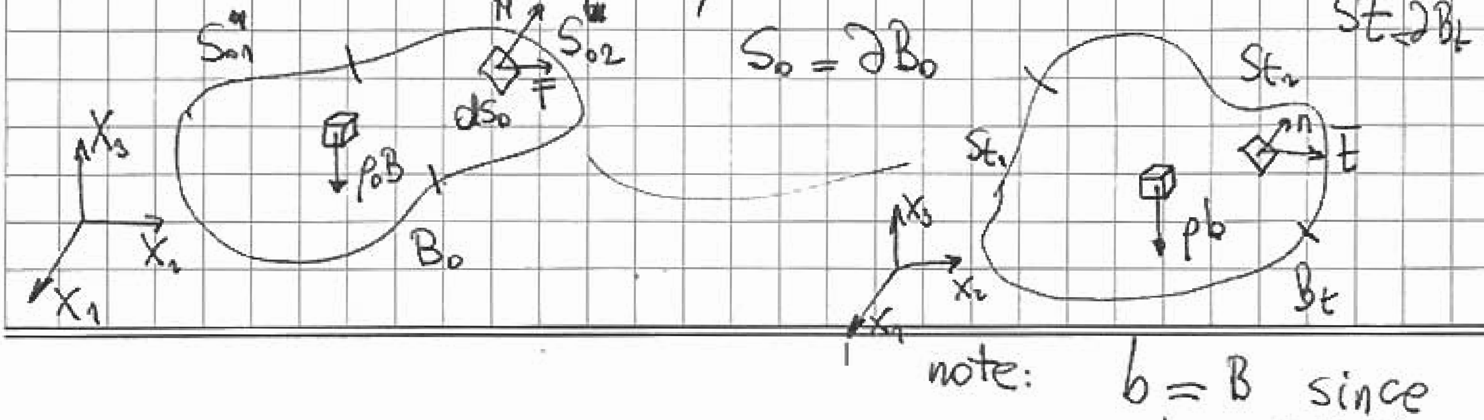
$P_{iJ} \dot{F}_{iJ}$ perfect diferential

$\Leftrightarrow \exists \; W(F) \equiv$ strain energy density (per unit undeformed volume)

$$\boxed{P_{iJ} = \frac{\partial W}{\partial F_{iJ}}}$$

Most general constitutive relations for nonlinear elastic solids.

Nonlinear elasticity BVP



note: $b = B$ since
$\rho dv_t b = \rho_o dv_o B_o$

Field eqs (Lagrangian setting)

1) Compatibility:

$$\left.\begin{aligned} F_{iJ} &= \varphi_{i,J} && \text{in } B_0 \\ \varphi_i &= \overline{\varphi_i} && \text{on } S_{01} \end{aligned}\right\}$$

2) Equilibrium:

$$\left.\begin{aligned} P_{iJ,J} + \rho_0 B_i &= \rho_0 A_i && \text{in } B_0 \\ P_{iJ} N_J &= \overline{T_i} && \text{on } S_{02} \end{aligned}\right\}$$

3) Constitutive:

$$P_{iJ} = \frac{\partial W(F)}{\partial F_{iJ}} \qquad \text{in } B_0$$