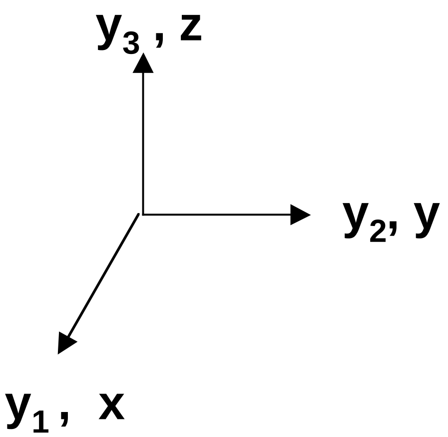
16.20 HANDOUT #2
Fall, 2002
Review of General Elasticity

NOTATION REVIEW (*e.g., for strain*)

Engineering		Contracted		Engineering "Tensor"		Tensor
ε_x	=	ε_1	=	ε_{xx}	=	ε_{11}
ε_y	=	ε_2	=	ε_{yy}	=	ε_{22}
ε_z	=	ε_3	=	ε_{zz}	=	ε_{33}
γ_{yz}	=	ε_4	=	$2\ \varepsilon_{yz}$	=	$2\ \varepsilon_{23}$
γ_{xz}	=	ε_5	=	$2\ \varepsilon_{xz}$	=	$2\ \varepsilon_{13}$
γ_{xy}	=	ε_6	=	$2\ \varepsilon_{xy}$	=	$2\ \varepsilon_{12}$

EQUATIONS OF ELASTICITY



Right-handed rectangular Cartesian coordinate system

15 equations/15 unknowns

1. Equilibrium (3)

$$\left.\begin{aligned}\frac{\partial\sigma_{11}}{\partial y_1} + \frac{\partial\sigma_{21}}{\partial y_2} + \frac{\partial\sigma_{31}}{\partial y_3} + f_1 = 0\\ \frac{\partial\sigma_{12}}{\partial y_1} + \frac{\partial\sigma_{22}}{\partial y_2} + \frac{\partial\sigma_{32}}{\partial y_3} + f_2 = 0\\ \frac{\partial\sigma_{13}}{\partial y_1} + \frac{\partial\sigma_{23}}{\partial y_2} + \frac{\partial\sigma_{33}}{\partial y_3} + f_3 = 0\end{aligned}\right\} \frac{\partial\sigma_{mn}}{\partial y_m} + f_n = 0$$

2. Strain-Displacement (6)

$$\left.\begin{array}{ll}
\varepsilon_{11} = \dfrac{\partial u_1}{\partial y_1} & \varepsilon_{21} = \varepsilon_{12} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial y_2} + \dfrac{\partial u_2}{\partial y_1}\right) \\
\varepsilon_{22} = \dfrac{\partial u_2}{\partial y_2} & \varepsilon_{31} = \varepsilon_{13} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial y_3} + \dfrac{\partial u_3}{\partial y_1}\right) \\
\varepsilon_{33} = \dfrac{\partial u_3}{\partial y_3} & \varepsilon_{32} = \varepsilon_{23} = \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial y_3} + \dfrac{\partial u_3}{\partial y_2}\right)
\end{array}\right\} \varepsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial y_n} + \frac{\partial u_n}{\partial y_m}\right)$$

3. Stress-Strain (6)

Generalized Hooke's Law: $\sigma_{mn} = E_{mnpq}\,\varepsilon_{pq}$

- Anisotropic:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1123} & 2E_{1113} & 2E_{1112} \\ E_{1122} & E_{2222} & E_{2233} & 2E_{2223} & 2E_{2213} & 2E_{2212} \\ E_{1133} & E_{2233} & E_{3333} & 2E_{3323} & 2E_{3313} & 2E_{3312} \\ E_{1123} & E_{2223} & E_{3323} & 2E_{2323} & 2E_{1323} & 2E_{1223} \\ E_{1113} & E_{2213} & E_{3313} & 2E_{1323} & 2E_{1313} & 2E_{1213} \\ E_{1112} & E_{2212} & E_{3312} & 2E_{1223} & 2E_{1213} & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}$$

- Orthotropic:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}$$

Compliance Form: $\varepsilon_{mn} = S_{mnpq}\,\sigma_{pq}$

where: $\underset{\sim}{E}^{-1} = \underset{\sim}{S}$

DEFINITION OF ENGINEERING CONSTANTS

1. Longitudinal (Young's) (Extensional) Moduli:

$$E_{mm} = \frac{\sigma_{mm}}{\varepsilon_{mm}}$$ due to σ_{mm} applied *only* *(no summation on m)*

2. Poisson's Ratios:

$$\nu_{nm} = -\frac{\varepsilon_{mm}}{\varepsilon_{nn}}$$ due to σ_{nn} applied *only* *(for $n \neq m$)*

Reciprocity: $\nu_{nm} E_m = \nu_{mn} E_n$ (no sum)
($m \neq n$)

3. Shear Moduli:

$$G_{mn} = \frac{\sigma_{mn}}{2\varepsilon_{mn}}$$ due to σ_{mn} applied *only*

(for m = 4, 5, 6)

(for $n \neq m$)

4. Coefficients of Mutual Influence:
(using contracted notation)

$$\eta_{mn} = \frac{-\varepsilon_n}{\varepsilon_m}$$ for σ_m applied *only*

(for m, n, = 1, 2, 3, 4, 5, 6, $m \neq n$)

(Note: one strain extensional, one strain shear)

Reciprocity here as well

5. Chentsov Coefficients:
(using contracted notation)

$$\eta_{mn} = \frac{-\varepsilon_n}{\varepsilon_m}$$ for σ_m applied *only*

(for m, n, = 4, 5, 6, $m \neq n$)

“ENGINEERING” STRESS-STRAIN EQUATIONS *(using contracted notation)*

$$\varepsilon_1 = \frac{1}{E_1}\left[\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3 - \eta_{14}\sigma_4 - \eta_{15}\sigma_5 - \eta_{16}\sigma_6\right]$$

$$\varepsilon_2 = \frac{1}{E_2}\left[- \nu_{21}\sigma_1 + \sigma_2 - \nu_{23}\sigma_3 - \eta_{24}\sigma_4 - \eta_{25}\sigma_5 - \eta_{26}\sigma_6\right]$$

$$\varepsilon_3 = \frac{1}{E_3}\left[-\nu_{31}\sigma_1 - \nu_{32}\sigma_2 + \sigma_3 - \eta_{34}\sigma_4 - \eta_{35}\sigma_5 - \eta_{36}\sigma_6\right]$$

$$\gamma_4 = \varepsilon_4 = \frac{1}{G_4}\left[-\eta_{41}\sigma_1 - \eta_{42}\sigma_2 - \eta_{43}\sigma_3 + \sigma_4 - \eta_{45}\sigma_5 - \eta_{46}\sigma_6\right]$$

$$\gamma_5 = \varepsilon_5 = \frac{1}{G_5}\left[-\eta_{51}\sigma_1 - \eta_{52}\sigma_2 - \eta_{53}\sigma_3 - \eta_{54}\sigma_4 + \sigma_5 - \eta_{56}\sigma_6\right]$$

$$\gamma_6 = \varepsilon_6 = \frac{1}{G_6}\left[-\eta_{61}\sigma_1 - \eta_{62}\sigma_2 - \eta_{63}\sigma_3 - \eta_{64}\sigma_4 - \eta_{65}\sigma_5 + \sigma_6\right]$$

In general:

$$\varepsilon_n = -\frac{1}{E_n}\sum_{m=1}^{6} \nu_{nm}\,\sigma_m$$

Note: $\nu_{nn} = -1$
and η’s --> ν’s

- Orthotropic form

In terms of ENGINEERING CONSTANTS (using *contracted notation*):

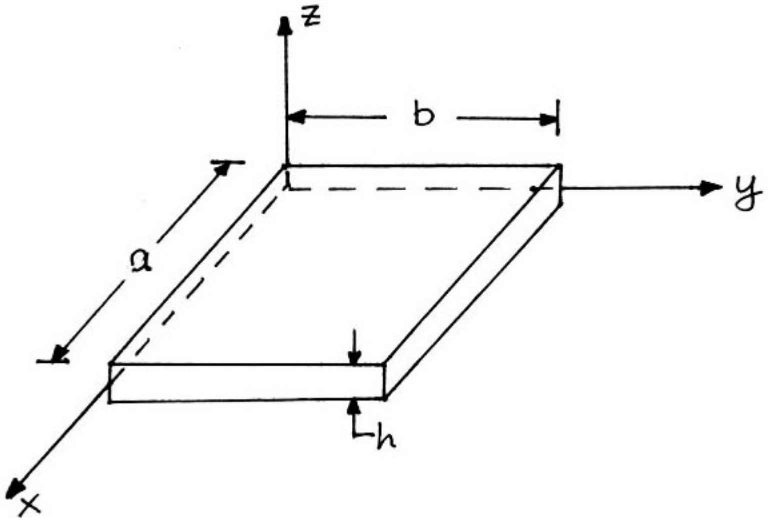
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_6} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

- Isotropic form

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

with: $G = \dfrac{E}{2(1 + \nu)}$

PLANE STRESS



$h << a, b$

$\sigma_{zz}, \sigma_{yz}, \sigma_{xz} = 0$

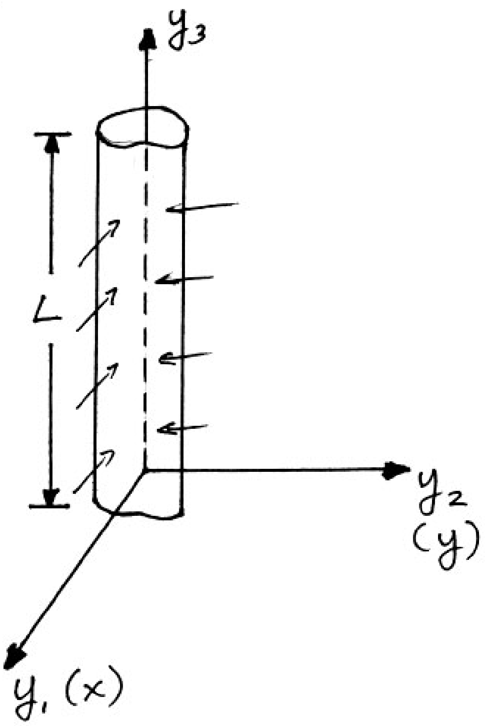
$$\frac{\partial}{\partial z} = 0$$

Anisotropic stress-strain equations

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{E_1}\left[\sigma_1 - \nu_{12}\sigma_2 - \eta_{16}\sigma_6\right] \\ \varepsilon_2 &= \frac{1}{E_2}\left[-\nu_{21}\sigma_1 + \sigma_2 - \eta_{26}\sigma_6\right] \\ \varepsilon_6 &= \frac{1}{G_6}\left[-\eta_{61}\sigma_1 - \eta_{62}\sigma_2 + \sigma_6\right] \end{aligned}\right\} \underline{\text{Primary}}$$

$$\left.\begin{aligned} \varepsilon_3 &= \frac{1}{E_3}\left[-\nu_{31}\sigma_1 - \nu_{32}\sigma_2 - \eta_{36}\sigma_6\right] \\ \varepsilon_4 &= \frac{1}{G_4}\left[-\eta_{41}\sigma_1 - \eta_{42}\sigma_2 - \eta_{46}\sigma_6\right] \\ \varepsilon_5 &= \frac{1}{G_5}\left[-\eta_{51}\sigma_1 - \eta_{52}\sigma_2 - \eta_{56}\sigma_6\right] \end{aligned}\right\} \underline{\text{Secondary}}$$

PLANE STRAIN



L >> x, y

$$\frac{\partial}{\partial z} = 0$$

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$$

SUMMARY

	Plane Stress	Plane Strain
Geometry:	thickness (y_3) << in-plane dimensions (y_1, y_2)	length (y_3) >> in-plane dimensions (y_1, y_2)
Loading:	$\sigma_{33} << \sigma_{\alpha\beta}$	$\sigma_{\alpha\beta}$ only $\partial/\partial y_3 = 0$
Resulting Assumptions:	$\sigma_{i3} = 0$	$\varepsilon_{i3} = 0$
Primary Variables:	$\varepsilon_{\alpha\beta,}$ $\sigma_{\alpha\beta}$, u_α	$\varepsilon_{\alpha\beta,}$ $\sigma_{\alpha\beta}$, u_α
Secondary Variable(s):	$\varepsilon_{33,}$ u_3	σ_{33}
Note:	Eliminate ε_{33} from eq. set by using $\sigma_{33} = 0$ σ - ε eq. and expressing ε_{33} in terms of $\varepsilon_{\alpha\beta}$	Eliminate σ_{33} from eq. Set by using σ_{33} σ - ε eq. and expressing σ_{33} in terms of $\varepsilon_{\alpha\beta}$

TRANSFORMATIONS

$$\tilde{\sigma}_{mn} = \ell_{\tilde{m}p} \, \ell_{\tilde{n}q} \, \sigma_{pq}$$

$$\tilde{\varepsilon}_{mn} = \ell_{\tilde{m}p} \, \ell_{\tilde{n}q} \, \varepsilon_{pq}$$

$$\tilde{x}_m = \ell_{\tilde{m}p} \, x_p$$

$$\tilde{u}_m = \ell_{\tilde{m}p} \, u_p$$

$$\tilde{E}_{mnpq} = \ell_{\tilde{m}r} \, \ell_{\tilde{n}s} \, \ell_{\tilde{p}t} \, \ell_{\tilde{q}u} \, E_{rstu}$$

where: $\ell_{\tilde{m}n}$ = cosine of angle from $\tilde{y}_m$ to y_n

OTHER COORDINATE SYSTEMS

$$F_1 (y_1, y_2, y_3) = \xi$$

$$F_2 (y_1, y_2, y_3) = \eta$$

$$F_3 (y_1, y_2, y_3) = \zeta$$

Example - Cylindrical Coordinates

$$\xi = r \qquad F_1 (y_1, y_2, y_3) = \sqrt{y_1^2 + y_2^2}$$

$$\eta = \theta \qquad F_2 (y_1, y_2, y_3) = \tan^{-1} \left(y_2 / y_1\right)$$

$$\zeta = z \qquad F_3 (y_1, y_2, y_3) = y_3$$

- Equilibrium:

$$r: \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0$$

$$\theta: \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta = 0$$

$$z: \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0$$

- (Engineering) Strain-Displacement:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}$$

$$\varepsilon_{zz} = \frac{\partial u_3}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}$$

$$\varepsilon_{\theta z} = \frac{1}{r} \frac{\partial u_3}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$

$$\varepsilon_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_3}{\partial r}$$

- (Isotopic) Stress-Strain:

$$\varepsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu\left(\sigma_{\theta\theta} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\left(\sigma_{rr} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right]$$

$$\varepsilon_{r\theta} = \frac{2(1 + \nu)}{E}\sigma_{r\theta}$$

$$\varepsilon_{\theta z} = \frac{2(1 + \nu)}{E}\sigma_{\theta z}$$

$$\varepsilon_{zr} = \frac{2(1 + \nu)}{E}\sigma_{zr}$$

STRESS FUNCTIONS

$$\nabla^4\phi = -E\alpha\nabla^2\left(\Delta T\right) - \left(1 - \nu\right)\nabla^2 V \quad \text{(isotropic)}$$

where: $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} + V$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} + V$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

EFFECTS OF THE ENVIRONMENT

Temperature

- Thermal Strain: $\varepsilon^T = \alpha \Delta T$

  α = Coefficient of Thermal Expansion (C.T.E.)

  general form: $\varepsilon_{ij}^T = \alpha_{ij} \Delta T$

- Total Strain = Mechanical Strain + Thermal Strain

$$\varepsilon_{ij} = \varepsilon_{ij}^M + \varepsilon_{ij}^T$$

$$\varepsilon_{ij}^M = S_{ijkl} \sigma_{kl}$$

- $\sigma_{kl} = E_{ijkl} \varepsilon_{ij} - E_{ijkl} \alpha_{ij} \Delta T$
- Transformation of α_{ij}:

$$\left.\begin{aligned}
\tilde{\alpha}_{11} &= \cos^2\theta \, \alpha_{11}^* + \sin^2\theta \, \alpha_{22}^* \\
\tilde{\alpha}_{22} &= \sin^2\theta \, \alpha_{11}^* + \cos^2\theta \, \alpha_{22}^* \\
\tilde{\alpha}_{12} &= \cos\theta \, \sin\theta \left(\alpha_{22}^* - \alpha_{11}^*\right)
\end{aligned}\right\}$$

α_{11}^*, α_{22}^* are C.T.E.'s in principal material axes

Sources of temperature differential

- Ambient environment
- Convection
- Aerodynamic heating

Adiabatic wall temp = $T_{AW} = \left[1 + \frac{\overset{\substack{\text{specific}\\\text{heat ratio}\\\searrow}}{\gamma} - 1}{2} \, \underset{\substack{\uparrow\\\text{recovery}\\\text{factor}}}{r} \, \overset{\substack{\text{Mach}\\\text{number}\\\downarrow}}{M_\infty^2}\right] T_\infty$

T_∞ = ambient temperature (°K)

heat flux: $q = \underset{\substack{\uparrow\\\text{heat transfer}\\\text{coefficient}}}{h} \left(T_{AW} - \underset{\substack{\uparrow\\\text{surface temperature}\\\text{of body}}}{T_s}\right)$

- Radiation
  - Emissivity

$$q = -\varepsilon\, \sigma\, T_s^4$$

q = heat flux
e = emissivity
σ = Stefan-Boltzman constant
T_s = surface temperature

  - Absorptivity

$$q = \alpha\, I_s\, \lambda$$

q = heat flux
α = absorptivity
I_s = intensity of source
λ = angle factor

- Conduction

$$q_i^T = -k_{ij}^T \frac{\partial T}{\partial x_j}$$

q_i^T = heat flux
k_{ij}^T = thermal conductivity

Fourier's equation:

$$\underset{\substack{\uparrow \\ \text{thermal} \\ \text{conductivity}}}{\frac{k_z^T}{\rho C}} \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial t}$$

Degradation of material properties

- Glass transition temperature
- $E(T)$, $\sigma_{ult}(T)$, $\sigma_y(T)$
- Creep

Other Environmental Effects

$$\varepsilon_{ij} = \varepsilon_{ij}^{M} + \Sigma\, \varepsilon_{ij}^{\varepsilon}$$

total = mechanical + ∑ environmental

- Moisture:

$$\varepsilon_{ij}^{s} = \beta_{ij}\, c \quad \left\{ \begin{array}{l} \varepsilon_{ij}^{S} = \text{swelling strain} \\ \beta_{ij} = \text{swelling coefficient} \\ c = \text{moisture concentration} \end{array} \right.$$

- General:

$$\varepsilon_{ij}{}^{E} = \chi_{ij}\, \chi \quad \left\{ \begin{array}{l} \varepsilon_{ij}^{E} = \text{environmental strain} \\ \chi_{ij} = \text{environmental operator} \\ \chi = \text{environmental scalar} \end{array} \right.$$

Piezoelectricity

- Piezoelectric strain:

$$\varepsilon_{ij}^{p} = d_{ijk}\, E_k$$

E_k = electric field
d_{ijk} = piezoelectric constant

- Coupled equations:

$$\sigma_{mn} = E_{mnij}\, \varepsilon_{ij} - E_{mnij}\, d_{ijk}\, E_k$$

$$D_i = e_{ik}\, E_k + d_{inm}\, \sigma_{mn}$$

e_{ik} = dielectric constant
D_i = electrical charge