Unit 4
Equations of Elasticity

<u>Readings</u>:

R	2.3, 2.6, 2.8
T & G	84, 85
B, M, P	5.1-5.5, 5.8, 5.9
	7.1-7.9
	6.1-6.3, 6.5-6.7
Jones	(as background on composites)

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Let's first review a bit …

from Unified, saw that there are 3 basic considerations in elasticity:

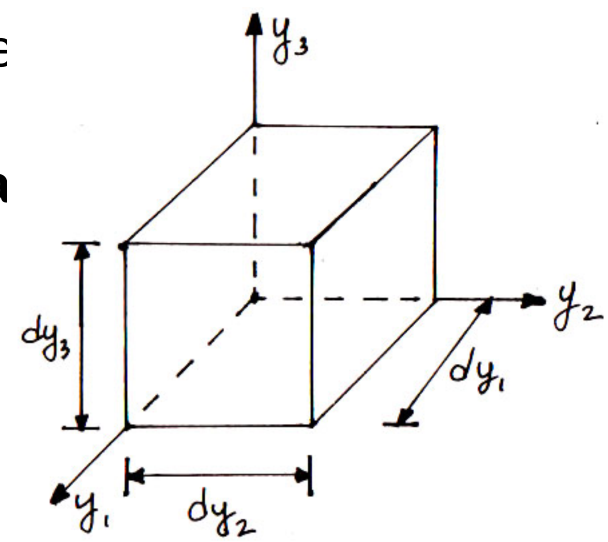
1. Equilibrium
2. Strain - Displacement
3. Stress - Strain Relations (<u>Constitutive</u> Relations)

Consider each:

1. <u>Equilibrium</u> (3)

- $\Sigma F_i = 0, \quad \Sigma M_i = 0$
- Free body diagrams
- Applying these to an infinitesimal eleme yields <u>3</u> equilibrium equations

Figure 4.1 **Representation of general infinitesima element**



$$\left.\begin{array}{ll}
\dfrac{\partial \sigma_{11}}{\partial y_1} + \dfrac{\partial \sigma_{21}}{\partial y_2} + \dfrac{\partial \sigma_{31}}{\partial y_3} + f_1 = 0 & \text{(4-1)} \\
\dfrac{\partial \sigma_{12}}{\partial y_1} + \dfrac{\partial \sigma_{22}}{\partial y_2} + \dfrac{\partial \sigma_{32}}{\partial y_3} + f_2 = 0 & \text{(4-2)} \\
\dfrac{\partial \sigma_{13}}{\partial y_1} + \dfrac{\partial \sigma_{23}}{\partial y_2} + \dfrac{\partial \sigma_{33}}{\partial y_3} + f_3 = 0 & \text{(4-3)}
\end{array}\right\}$$

$$\boxed{\frac{\partial \sigma_{mn}}{\partial y_m} + f_n = 0}$$

2. <u>Strain - Displacement</u> (6)

- Based on geometric considerations
- Linear considerations (I.e., *small* strains only -- we will talk about large strains later)

(and *<u>infinitesimal</u>* displacements only)

$$\left.\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial y_1} && \text{(4-4)}\\
\varepsilon_{22} &= \frac{\partial u_2}{\partial y_2} && \text{(4-5)}\\
\varepsilon_{33} &= \frac{\partial u_3}{\partial y_3} && \text{(4-6)}\\
\varepsilon_{21} = \varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1}\right) && \text{(4-7)}\\
\varepsilon_{31} = \varepsilon_{13} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial y_3} + \frac{\partial u_3}{\partial y_1}\right) && \text{(4-8)}\\
\varepsilon_{32} = \varepsilon_{23} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial y_3} + \frac{\partial u_3}{\partial y_2}\right) && \text{(4-9)}
\end{aligned}\right\}$$

$$\boxed{\varepsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial y_n} + \frac{\partial u_n}{\partial y_m}\right)}$$

3. <u>Stress - Strain</u> (6)

$$\boxed{\sigma_{mn} = E_{mnpq}\, \varepsilon_{pq}}$$

we'll come back to this …

Let's review the "4th important concept":
<u>Static Determinance</u>

There are there possibilities (as noted in U.E.)

a. A structure is not sufficiently restrained

(fewer reactions than d.o.f.)

$\hookrightarrow$ degrees of freedom

$\Rightarrow$ DYNAMICS

b. Structure is exactly (or "simply") restrained

(# of reactions = # of d.o.f.)

$\Rightarrow$ STATICS (statically determinate)

<u>Implication</u>: can calculate stresses via equilibrium (as done in Unified)

c. Structure is overrestrained

(# reactions > # of d.o.f.)

⇒STATICALLY INDETERMINATE

…must solve for reactions <u>simultaneously</u> with stresses, strains, etc.

in this case, you must employ the stress-strain equations

--> Overall, this yields for elasticity:

<u>15 unknowns</u>	and	<u>15 equations</u>
6 strains = ε_{mn}		3 equilibrium (σ)
6 stresses = σ_{mn}		6 strain-displacements (ε)
3 displacements = u_m		6 stress-strain (σ - ε)

<u>IMPORTANT POINT:</u>

The first two sets of equations are "universal" (independent of the material) as they depend on geometry (strain-displacement) and equilibrium (equilibrium). <u>Only</u> the stress-strain equations are dependent on the material.

<u>One</u> <u>other</u> <u>point</u>: Are all these equations/unknowns independent? <u>NO</u>

Why? --> Relations between the strains and displacements (due to geometrical considerations result in the <u>Strain</u> <u>Compatibility</u> <u>Equations</u>

(as you saw in Unified)

General form is:

$$\boxed{\frac{\partial^2 \varepsilon_{nk}}{\partial y_m \partial y_\ell} + \frac{\partial^2 \varepsilon_{m\ell}}{\partial y_n \partial y_k} - \frac{\partial^2 \varepsilon_{n\ell}}{\partial y_m \partial y_k} - \frac{\partial^2 \varepsilon_{mk}}{\partial y_n \partial y_\ell} = 0}$$

This results in <u>6</u> strain-compatibility (in 3-D).
What a mess!!!

What do these really tell us???

The strains must be <u>compatible</u>, they cannot be prescribed in an arbitrary fashion.

Let's consider an **example:**

<u>Step 1</u>: consider how shear strain (ε_{12}) is related to displacement:

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1}\right)$$

Note that deformations (u_m) must be continuous single-valued functions for continuity. (or it doesn't make <u>physical</u> <u>sense</u>!)

<u>Step 2</u>: Now consider the case where there are gradients in the strain field

$$\frac{\partial \varepsilon_{12}}{\partial y_1} \neq 0, \quad \frac{\partial \varepsilon_{12}}{\partial y_2} \neq 0$$

This is the most general case and most likely in a general structure

Take derivatives on both sides:

$$\Rightarrow \quad \frac{\partial^2 \varepsilon_{12}}{\partial y_1 \partial y_2} = \frac{1}{2}\left(\frac{\partial^3 u_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 u_2}{\partial y_1^2 \partial y_2}\right)$$

<u>Step 3</u>: rearrange slightly and recall other strain-displacement equations

$$\frac{\partial u_1}{\partial y_1} = \varepsilon_1, \quad \frac{\partial u_2}{\partial y_2} = \varepsilon_2$$

$$\Rightarrow \quad \frac{\partial^2 \varepsilon_{12}}{\partial y_1 \partial y_2} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_{11}}{\partial y_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial y_1^2}\right)$$

So, the gradients in strain are related in certain ways since they are all related to the 3 displacements.

Same for other 5 cases …

Let’s now go back and spend time with the …

Stress-Strain Relations and the Elasticity Tensor

In Unified, you saw particular examples of this, but we now want to generalize it to encompass all cases.

The basic relation between force and displacement (recall 8.01) is Hooke’s Law:

$F = kx$

k – spring constant (linear case)

If this is extended to the three-dimensional case and applied over infinitesimal areas and lengths, we get the relation between stress and strain known as:

Generalized Hooke's law:

$$\boxed{\sigma_{mn} = E_{mnpq}\, \varepsilon_{pq}}$$

where E_{mnpq} is the "elasticity tensor"

How many components does this appear to have?

$m, n, p, q = 1, 2, 3$

$\Rightarrow 3 \times 3 \times 3 \times 3 = 81$ components

But there are several symmetries:

1. Since $\sigma_{mn} = \sigma_{nm}$ (energy considerations)

$\Rightarrow E_{mnpq} = E_{nmpq}$

(symmetry in switching first two indices)

2. Since $\varepsilon_{pq} = \varepsilon_{qp}$ (geometrical considerations)

$\Rightarrow E_{mnpq} = E_{mnqp}$

(symmetry in switching <u>last two indices</u>)

3. From thermodynamic considerations

(1st law of thermo)

$\Rightarrow E_{mnpq} = E_{pqmn}$

(symmetry in switching <u>pairs of indices</u>)

Also note that:

Since $\sigma_{mn} = \sigma_{nm}$, the apparent 9 equations for stress are only 6!

With these symmetrics, the resulting equations are:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1123} & 2E_{1113} & 2E_{1112} \\ E_{1122} & E_{2222} & E_{2233} & 2E_{2223} & 2E_{2213} & 2E_{2212} \\ E_{1133} & E_{2233} & E_{3333} & 2E_{3323} & 2E_{3313} & 2E_{3312} \\ E_{1123} & E_{2223} & E_{3323} & 2E_{2323} & 2E_{1323} & 2E_{1223} \\ E_{1113} & E_{2213} & E_{3313} & 2E_{1323} & 2E_{1313} & 2E_{1213} \\ E_{1112} & E_{2212} & E_{3312} & 2E_{1223} & 2E_{1213} & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}$$

Results in <u>21</u> independent components of the elasticity tensor

- Along diagonal (6)
- Upper right half of matrix (15)

[don't worry about 2's]

<u>Also</u> *<u>note</u>*: 2's come out automatically…don't put them in $\underset{\sim}{\varepsilon}$

For example: $\sigma_{12} = \ldots E_{1212}\, \varepsilon_{12} + E_{1221}\, \varepsilon_{21} \ldots$
$$= \ldots 2E_{1212}\, \varepsilon_{12} \ldots$$

These E_{mnpq} can be placed into <u>3 groups</u>:

- Extensional strains to extensional stresses

$$\begin{matrix} E_{1111} & E_{1122} \\ E_{2222} & E_{1133} \\ E_{3333} & E_{2233} \end{matrix}$$

e.g., $\sigma_{11} = \ldots E_{1122}\, \varepsilon_{22} \ldots$

- Shear strains to shear stresses

$$\begin{matrix} E_{1212} & E_{1213} \\ E_{1313} & E_{1323} \\ E_{2323} & E_{2312} \end{matrix}$$

$$\text{e.g.,} \quad \sigma_{12} = \ldots\ 2E_{1223}\ \varepsilon_{23}\ \ldots$$

- Coupling term: extensional strains to shear stress *or* shear strains to extensional stresses

$$\begin{matrix} E_{1112} & E_{2212} & E_{3312} \\ E_{1113} & E_{2213} & E_{3313} \\ E_{1123} & E_{2223} & E_{3323} \end{matrix}$$

e.g.,

$$\sigma_{12} = \ldots E_{1211}\ \varepsilon_{11} \ldots$$

$$\sigma_{11} = \ldots 2E_{1123}\ \varepsilon_{23} \ldots$$

A material which behaves in this manner is "fully" anisotropic

However, there are currently no useful engineering materials which have 21 different and independent components of E_{mnpq}

The "type" of material (with regard to elastic behavior) dictates the number of *independent* components of E_{mnpq}:

Material Type	# of Independent Components of E_{mnpq}
Anisotropic	21
Monoclinic	13
Orthotropic	9
Tetragonal	6
"Transversely Isotropic"*	5
Cubic	3
Isotropic	2

Composite Laminates

Useful Engineering Materials

Basic Composite Ply

Metals (on average)

<u>Good Reference</u>: BMP, Ch. 7

**not in BMP*

For <u>orthotropic</u> <u>materials</u> (which is as complicated as we usually get), there are no coupling terms in the *<u>principal axes of the material</u>*

- When you apply an extensional stress, no shear strains arise

  e.g., $E_{1112} = 0$

  (total of 9 terms are now zero)

- When you apply a shear stress, no extensional strains arise

  (some terms become zero as for previous condition)

- Shear strains (stresses) in one plane do not cause shear strains (stresses) in another plane

  $(E_{1223} , E_{1213} , E_{1323} = 0)$

With these additional terms zero, we end up with 9 independent components:

(21 - 9 - 3 = 9)

and the equations are:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}$$

For other cases, no more terms become zero, but the terms are not Independent.

For example, for isotropic materials:

- $E_{1111} = E_{2222} = E_{3333}$
- $E_{1122} = E_{1133} = E_{2233}$
- $E_{2323} = E_{1313} = E_{1212}$
- And there is one other equation relating E_{1111} , E_{1122} and E_{2323}

$\Rightarrow$ 2 independent components of E_{mnpq}

(we'll see this more when we do engineering constants)

Why, then, do we bother with anisotropy?

<u>Two</u> <u>reasons</u>:

1. Someday, we may have useful fully anisotropic materials (certain crystals now behave that way) Also, 40-50 years ago, people only worried about isotropy

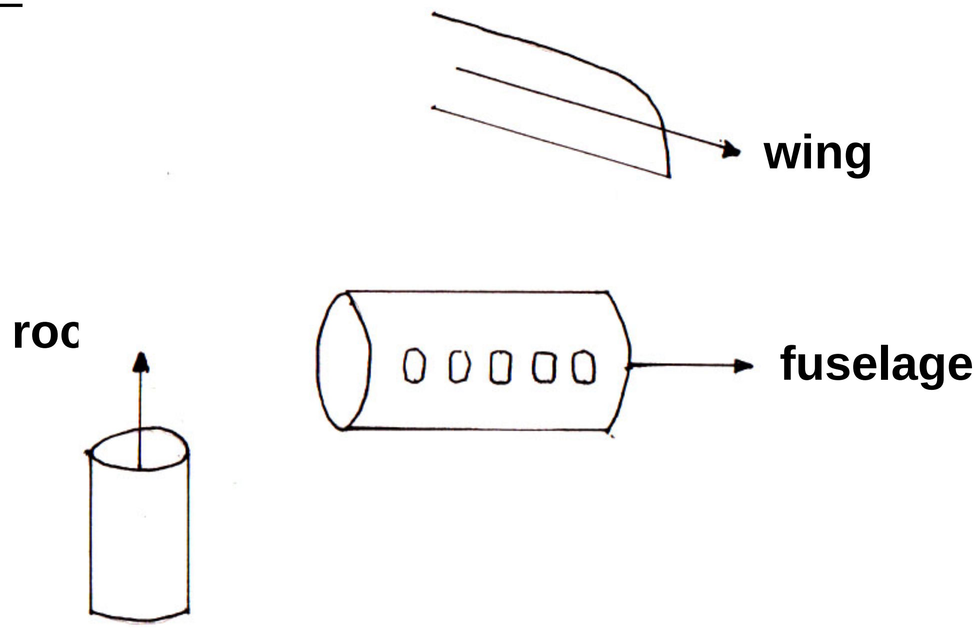
2. It may not always be convenient to describe a structure (i.e., write the governing equations) along the principal material axes.

   How else?

   <u>Loading</u> <u>axes</u>

<u>Examples</u>

Figure 4-2

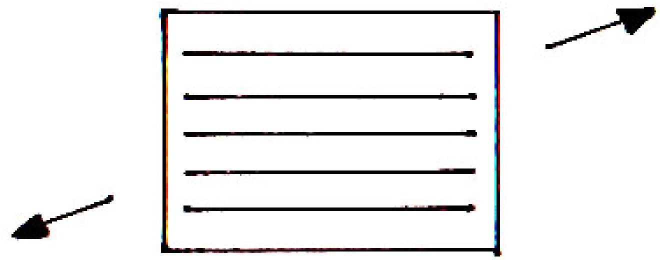


In these other axis systems, the material may have "more" elastic components. **But it really doesn't.**

(you can't "create" elastic components just by describing a material in a different axis system, the inherent properties of the material stay the same).

Figure 4-3 **Example: Unidirectional composite (transversely isotropic)**

No shear / extension coupling



Shears with regard to loading axis but still no inherent shear/extension coupling

In order to describe full behavior, need to do …TRANSFORMATIONS

(we'll review this/expand on it later)

--> It is often useful to consider the relationship between stress and strain (opposite way). For this we use

COMPLIANCE

$$\varepsilon_{mn} = S_{mnpq}\ \sigma_{pq}$$

where: S_{mnpq} = compliance tensor

Using matrix notation:

$\underset{\sim}{\sigma} = \underset{\approx}{E}\, \underset{\sim}{\varepsilon}$

and $\underset{\approx}{E}^{-1}\, \underset{\sim}{\sigma} = \underset{\sim}{\varepsilon}$

inverse

with $\underset{\sim}{\varepsilon} = \underset{\approx}{S}\, \underset{\sim}{\sigma}$

this means

$$\boxed{\underset{\approx}{E}^{-1} = \underset{\approx}{S}}$$

$\Rightarrow \underset{\approx}{E}\, \underset{\approx}{S} = \underset{\approx}{I}$

$\Rightarrow$The compliance matrix is the inverse of the elasticity matrix

Note: the same symmetries apply to S_{mnpq} as to E_{mnpq}

Meaning of each:

- Elasticity term E_{mnpq}: amount of stress (σ_{mn}) related to the deformation/strain (ε_{pq})
- Compliance term S_{mnpq}: amount of strain (ε_{mn}) the stress (σ_{pq}) causes

These are useful in defining/ determining the "engineering constants" which we will review / introduce / expand on in the next lecture.

All of this presentation on elasticity (and what you had in Unified) is based on assumptions which limit their applicability:

- Small strain
- Small displacement / infinitesimal (linear) strain

Fortunately, most engineering structures are such that these assumptions cause negligible error.

However, there are cases where this is not true:

- Manufacturing (important to be able to convince)
- Compliant materials
- Structural examples: dirigibles, …

So let's explore:

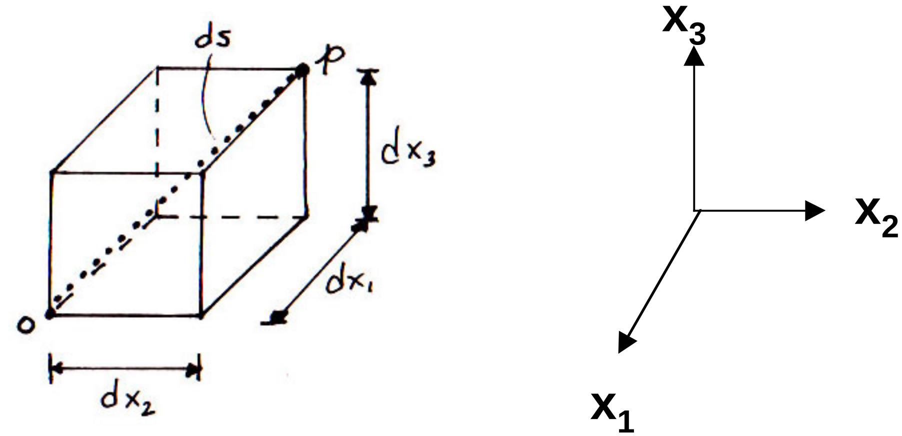
Large strain and the formal definition of strain

What we defined before are the physical manifestation of strain / deformation

- Relative elongation
- Angular rotation

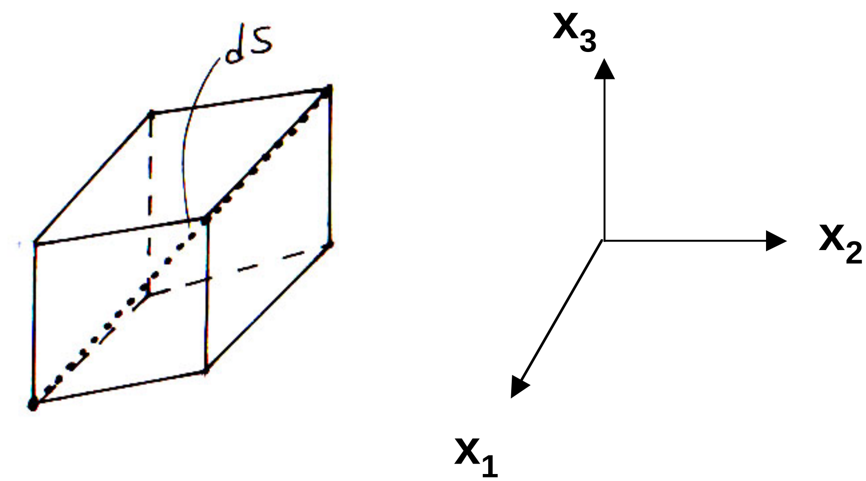
Strain is formally defined by considering the diagonal length of a cube:

Figure 4-4 **undeformed (small letters)**



and looking at the change in length under general (and possibly large) deformation:

Figure 4-5 **deformed** **(capital letters)**



The formal definition of the strain tensor is:

$$\boxed{2\gamma_{mn}\, dx_m\, dx_n = (dS)^2 - (ds)^2}$$

$$\Rightarrow 2\gamma_{11}\, dx_1\, dx_1 + 2\gamma_{22}\, dx_2\, dx_2 + 2d\gamma_{33}\, dx_3\, dx_3$$

$$+ 2(\gamma_{12} + \gamma_{21})\, dx_1\, dx_2 + 2(\gamma_{13} + \gamma_{31})\, dx_1\, dx_3$$

$$+ 2(\gamma_{23} + \gamma_{32})\, dx_2\, dx_3 = (dS)^2 - (ds)^2$$

where γ_{mn} = formal strain tensor.

This is a **definition**. The physical interpretation is related to this but not directly in the general case.

One can show (see BMP 5.1 - 5.4) that the formal strain tensor is related to relative elongation (the familiar $\frac{\Delta \ell}{\ell}$) via:

relative elongation in m-direction:

$$E_m = \sqrt{1 + 2\gamma_{mm}} - 1 \qquad \text{(no summation on m)}$$

and is related to angular change via:

$$\sin\phi_{mn} = \frac{2\gamma_{mn}}{(1 + E_m)(1 + E_n)}$$

Thus, it also involves the relative elongations!

Most structural cases deal with relatively small strain. If the relative elongation is small (<<100%)

$$\Rightarrow E_m << 1$$

look at:

$$E_m = \sqrt{1 + 2\gamma_{mm}} - 1$$

$$\Rightarrow (E_m + 1)^2 = 1 + 2\gamma_{mm}$$

$$E_m^2 + 2E_m = 2\gamma_{mm}$$

but if $E_m << 1$, then $E^2_m \approx 0$

$$\Rightarrow E_m = \gamma_{mm}$$

Relative elongation = strain

$$\frac{\Delta \ell}{\ell} = \varepsilon$$ *small strain approximation!*

Can assess this effect by comparing $2E_m$ and $E_m\ (2 + E_m)$

relative elongation = E_m	**$2E_m$**	**$E_m\ (2 + E_m)$**	**% error**
0.01	0.02	0.0201	0.5%
0.02	0.04	0.0404	1.0%
0.05	0.10	0.1025	2.4%
0.10	0.20	0.2100	4.8%

Similarly, consider the general expression for rotation:

$$\sin\phi_{mn} = \frac{2\gamma_{mn}}{\left(1 + E_m\right)\left(1 + E_n\right)}$$

for small elongations ($E_m << 1$, $E_n << 1$)

$$\Rightarrow \sin\phi_{mn} = 2\gamma_{mn}$$

<u>and</u>, if the rotation is small:

$$\sin\phi_{mn} \approx \phi_{mn}$$

$$\Rightarrow \phi_{mn} = 2\gamma_{mn}$$

$$= 2\varepsilon_{mn}$$ *small strain approximation!* (as before)

<u>Note</u>: factor of 2 !

Even for a balloon, the small strain approximation may be good enough

<u>So</u>: from now on, small strain assumed, but

- understand limitations
- be prepared to deal with large strain
- know difference between formal definition and the engineering approximation which relates dire<u>ctly</u> to physical reality.

What is the other limitation? It deals with displacement, so consider

Large Displacement and Non-Infinitesimal (Non-linear) Strain

See BMP 5.8 and 5.9

The general strain-displacement relation is:

$$\gamma_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} + \frac{\partial u_r}{\partial x_m}\frac{\partial u_s}{\partial x_n}\delta_{rs}\right)$$

Where:

δ_{rs} = Kronecker delta

The latter terms are important for larger displacements but are higher order for small displacements and can then be ignored to arrive back at:

$$\varepsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial y_n} + \frac{\partial u_n}{\partial y_m}\right)$$

How to assess?

Look at

$$\frac{\partial u_m}{\partial x_n} \quad \text{vs.} \quad \frac{\partial u_r}{\partial x_m} \, \frac{\partial u_s}{\partial x_n} \, \delta_{rs}$$

and compare magnitudes

Small vs. large and linear vs. nonlinear will depend on:

- material(s)
- structural configuration
- mode of behavior
- the loading

Examples

- Rubber in inflated structures
  $\Rightarrow$ Large strain (Note: generally means larger displacement)

- Diving board of plastic or wood
  $\Rightarrow$ Small strain but possibly large displacement (will look at this more when we deal with beams)

- Floor beam of steel

  $\Rightarrow$ Small strain and linear strain (Note: linear strain must also be
  $\Rightarrow$ small)

Next…back to constitutive constants…now their physical reality