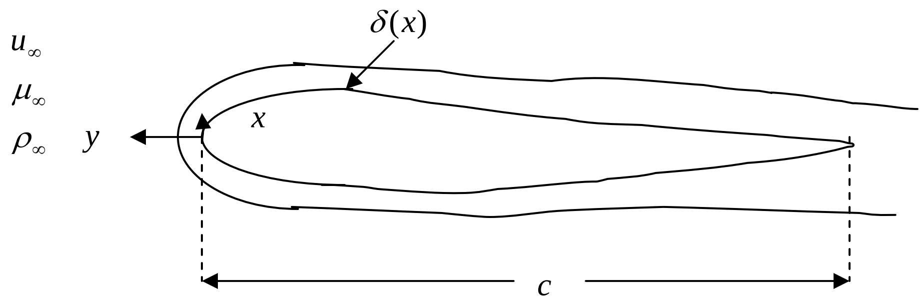
Laminar Boundary Layer Order of Magnitude Analysis



<u>Assumptions</u> (in addition to incompressible, steady & 2-D)

* Changes in $x-$direction occur over a distance c
  $\Rightarrow \dfrac{\partial}{\partial x} \sim \dfrac{1}{C}$ or, we write $\dfrac{\partial}{\partial x} = O(\dfrac{1}{C})$
* Changes in $y-$direction occur over a distance δ
  $\Rightarrow \dfrac{\partial}{\partial y} \sim \dfrac{1}{\delta}$ or, $\dfrac{\partial}{\partial y} = O(\dfrac{1}{\delta})$
* $\delta << C$ (boundary layer is thin)
* $\mathrm{Re} = \dfrac{\rho_\infty u_\infty c}{\mu_\infty} >> 1$ (Reynolds number is large)
* Changes in $x-$velocity are proportional to u_∞
  $\Delta u = O(u_\infty)$
* Changes in pressure are proportional to $\rho_\infty u_\infty^2$
  $\Delta p = O(\rho_\infty u_\infty^2)$

Now, let's do the order of magnitude analysis:

<u>Conservation of Mass</u>

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Now, we introduce our assumed orders. For example:

$$\frac{\partial u}{\partial x} = O(\frac{u_\infty}{c})$$

Since we have not assumed an order for v variations, just leave this as simply Δv and say,

$$\frac{\partial v}{\partial y} = \mathrm{O}(\frac{\Delta v}{\delta})$$

Since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, they obviously have equal (though opposite) magnitudes and therefore orders. Thus, the changes in v scale as:

$$\Delta v = \mathrm{O}(u_\infty \frac{\delta}{C})$$

Since $v = 0$ at a wall, $\Delta v = \mathrm{O}(v)$ itself and we can say

$$\boxed{\frac{v}{u_\infty} = \mathrm{O}(\frac{\delta}{c})}$$

$x-$momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\mathrm{O}(\rho_\infty \frac{u_\infty^2}{c})\ \mathrm{O}(\rho_\infty \frac{u_\infty^2}{c})\ \mathrm{O}(\rho_\infty \frac{u_\infty^2}{c})\ \mathrm{O}(\frac{\mu_\infty u_\infty}{c^2})\ \mathrm{O}(\frac{\mu_\infty u_\infty}{\delta^2})$$

If we compare the last two terms, clearly

$$\frac{\mu_\infty u_\infty}{c^2} << \frac{\mu_\infty u_\infty}{\delta^2}$$

Since we have assumed $\delta << C$. Thus, we can assume that $\mu \frac{\partial^2 u}{\partial y^2}$ is small.

In order for any viscous terms to remain, this requires that $\mu \frac{\partial^2 u}{\partial x^2}$ be equal in order to the remaining inviscid terms. That is, e.g

$$\mathrm{O}(\rho u \frac{\partial u}{\partial x}) = \mathrm{O}(\mu \frac{\partial^2 u}{\partial y^2})$$

$$\Rightarrow \mathrm{O}(\frac{\rho_\infty u_\infty^2}{c}) = \mathrm{O}(\frac{\mu_\infty u_\infty}{\delta^2})$$

$$\Rightarrow (\frac{\delta}{c})^2 = \mathrm{O}(\frac{\mu_\infty}{\rho_\infty u_\infty c})$$

In other words,

$$\boxed{\frac{\delta}{C} = \mathrm{O}(\frac{1}{\sqrt{\mathrm{Re}}})} \qquad \Leftarrow \text{Major result!}$$

$y-$momentum

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2}$$

$$\mathrm{O}(\rho_\infty u_\infty^2 \frac{\delta}{c^2}) \ \mathrm{O}(\rho_\infty u_\infty^2 \frac{\delta}{c^2}) \ \mathrm{O}(\frac{\rho_\infty u_\infty^2}{\delta}) \ \mathrm{O}(\mu_\infty u_\infty \frac{\delta}{c^3}) \ \mathrm{O}(\mu_\infty u_\infty \frac{1}{\delta c})$$

Normalizing all of these orders by $\frac{1}{\rho_\infty u_\infty^2 / C}$ and substituting in for $\mathrm{Re} = \frac{\rho_\infty u_\infty C}{\mu_\infty}$ and $\frac{\delta}{C} = \mathrm{O}(\frac{1}{\sqrt{\mathrm{Re}}})$ gives:

$$\mathrm{O}(\frac{1}{\sqrt{\mathrm{Re}}}) \ \mathrm{O}(\frac{1}{\sqrt{\mathrm{Re}}}) \ \mathrm{O}(\sqrt{\mathrm{Re}}) \ \mathrm{O}(\frac{1}{\mathrm{Re}^{3/2}}) \ \mathrm{O}(\frac{1}{\sqrt{\mathrm{Re}}})$$

Clearly, except for $\frac{\partial p}{\partial y}$, all other terms are small as Re increases. Thus, we must conclude that:

$$\frac{\partial p}{\partial y} \text{ is small}$$

$$\Rightarrow \boxed{0 = \frac{\partial p}{\partial y}} \text{ is the y} - \text{momentum equation for a boundary layer}$$