M5 Simple Beam Theory (continued)

In the previous lecture we had reached the point of obtaining 5 equations, 5 unknowns by application of equations of elasticity and modeling assumptions for beams (plane sections remain plane and perpendicular to the mid plane):

$$u(x,y,z) = -z\frac{dw}{dx} \tag{1}$$

$$w(x,y,z) = w(x) \tag{2}$$

$$\varepsilon_{xx} = -z\frac{d^2w}{dx^2} \tag{3}$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} \tag{4}$$

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{zx}}{\partial z} = 0 \tag{5}$$

5 eqns, 5 unknowns : $w, u, \varepsilon_{xx}, \sigma_{xx}, \sigma_{xz}$

Solutions: Stresses & deflections

First consider relationship between stresses and internal force and moment resultants for beams (axial force, F, shear force, S, and bending moment M). Consider rectangular cross section (will generalize for arbitrary cross-section shortly)
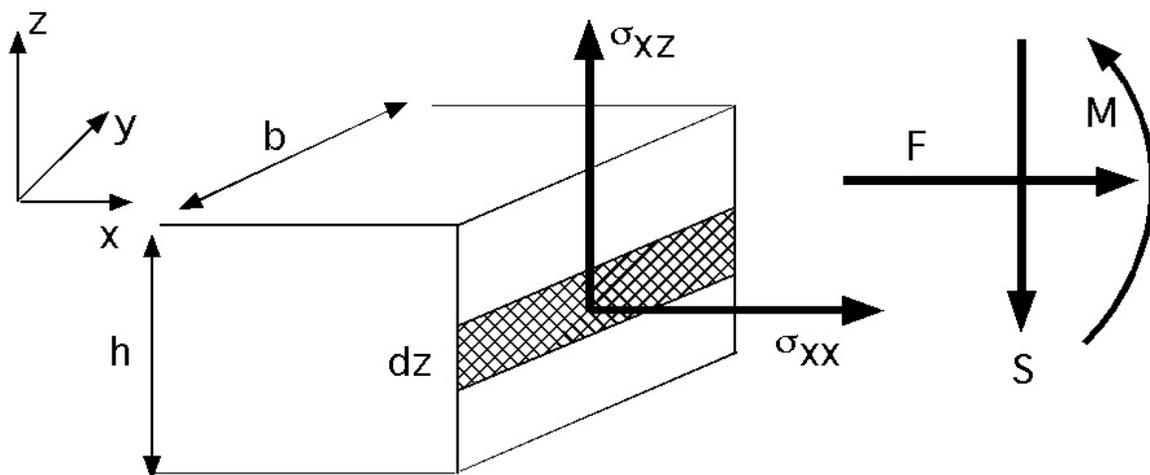
Resultant forces and moments are related to the stresses via considerations of equipollence:

$$F = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{xx} b\, dz \tag{6}$$

$$S = -\int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{xz} b\, dz \tag{7}$$

$$M = -\int\limits_{-h/2}^{h/2} \sigma_{xx} bz\, dz \tag{8}$$

Now we combine and substitute between the equations derived above.
combine (3) and (4) to obtain:

$$\sigma_{xx} = E_x \varepsilon_{xx} = -E_x z \frac{d^2 w}{dx^2}$$

Now substitute this in (7)

$$F = -E_x \frac{d^2 w}{dx^2} \int\limits_{-h/2}^{h/2} zb\, dz = \left[-E_x \frac{d^2 w}{dx^2} \frac{z^2}{2} b \right]_{-h/2}^{h/2} = 0$$

Which is correct, since there is no axial force in pure beam case.

Similarly in (8)

$$M = E_x \frac{d^2 w}{dx^2} \int\limits_{\frac{-h}{2}}^{\frac{+h}{2}} z^2 b\, dz \ .$$

We can write this more succinctly as:

$$M = E_x I \underbrace{\frac{d^2 w}{dx^2}}_{\text{curvature}}$$

this is the moment curvature relation - positive moment results in positive curvature (upward). The quantity "EI" defines the stiffness of beam – sometimes called "flexural rigidity" – note that it depends on both the shape and the material property, E.

Where $I = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 b\,dz$ is the *second moment of area* (similar to moment of inertia for

dynamics – see 8.01)

Recall that $\sigma_{xx} = -E_x z\dfrac{d^2 w}{dx^2} = -E_x z\dfrac{M}{E_x I}$

$$\boxed{\sigma_{xx} = -\frac{Mz}{I}}$$

This is the moment - stress relationship

Implies a linear variation of stress,

Maximum stress at edges (top and bottom) of beam

More general definition of I :

See Crandall, Dahl and Lardner 7.5

For general, symmetric, cross-section
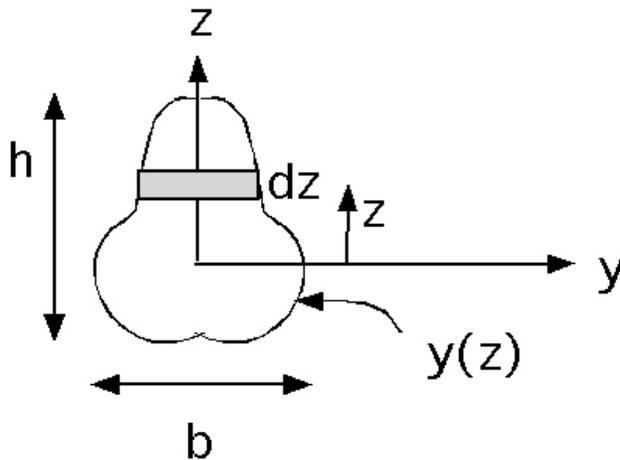
$I_{yy} = \int_A z^2\, dA = \int_{-h/2 - y(z)}^{h/2 + y(z)} \int z^2\, dy\,dz$ I_{yy} is second moment of area about y axis.

Note that there is also a moment of inertia about the z axis – I_{zz}. In general we will be concerned about the possibility of bending about multiple axes, and the bending of non-

symmetric cross-section. However, we will not consider these cases in Unified (see 16.20)

Note, for particular case of a rectangular section $I = \dfrac{1}{12} bh^3$

$$\underbrace{}_{[L^4]}$$

We need to define centroid: center of area (analogous to center of mass). Position calculated by taking moments of area about arbitrary axis. Hence z position given by:

$$z_{\text{Centroid}} = \dfrac{\displaystyle\int_A z\,dA}{\displaystyle\int_A dA} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \text{No net moment about centroid}$$

In bending centroid is unstrained – also known as the "neutral axis" or "neutral plane"

Can also use parallel axis theorem to calculate second moments of area

$$I_{\text{total}} = \sum I_i + \sum A_i d_i^2$$

Second moment of area about centroid of area	area	Distance of centroid from total section centroid
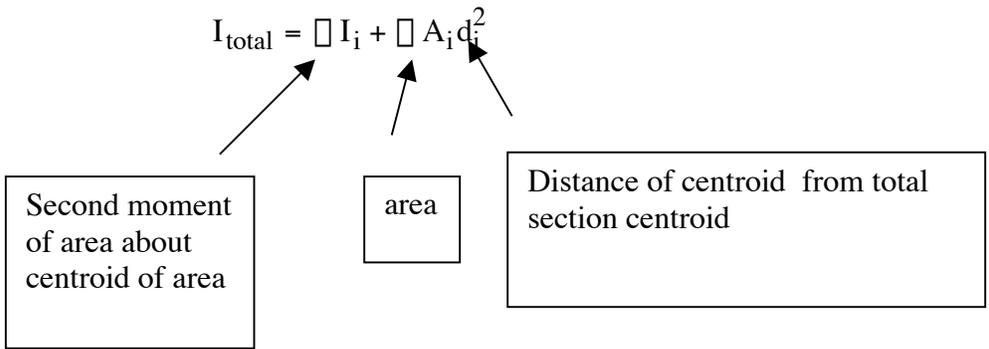
Useful to simplify problems, e.g. for I beam cross-section (structurally efficient form, effectively used as spars in many aircraft).

Flange (spar cap)

y ———— y

Web
Spar

Flange   is equivalent to:
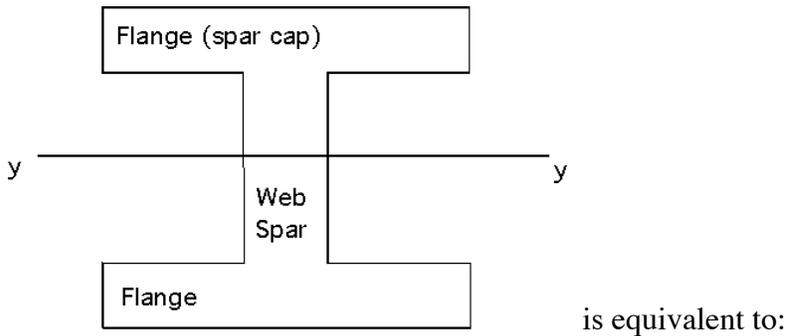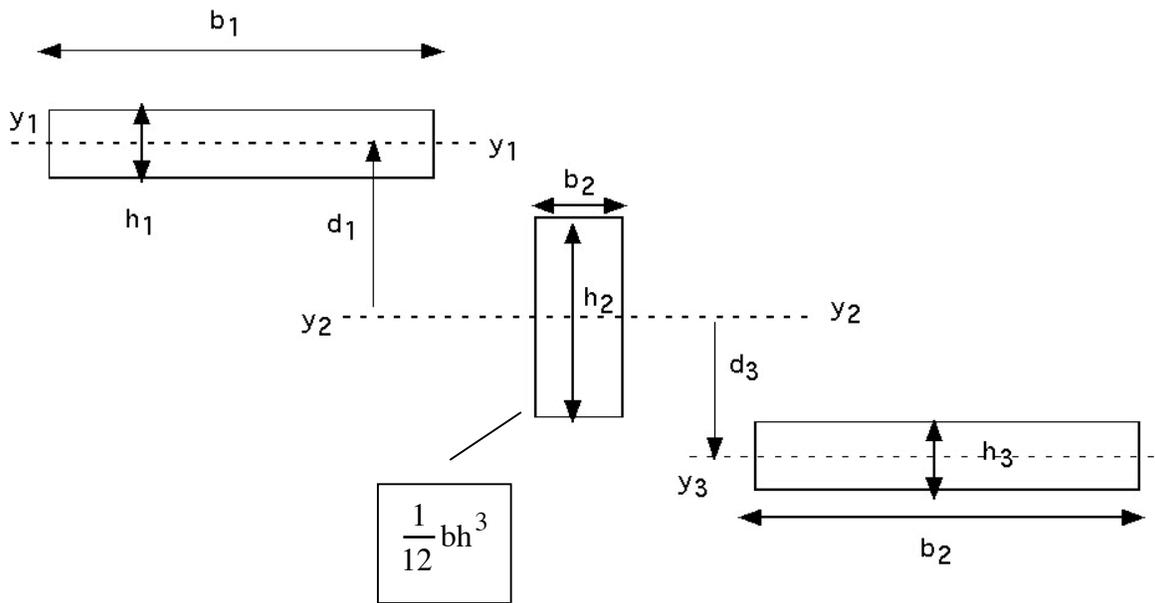
$$I_{\,total} = I_{y_1 y_1} + A_1 d_1^2 + I_{y_2 y_2} + I_{y_3 y_3} + A_3 d_3^2$$

Changing area distribution by moving material from the web to the flanges increases section efficiency measured as I/A - resistance to bending for a given amount of material used.

M6 Shear Stresses in Simple Beam Theory

Reading: Crandall, Dahl and Lardner 7.6

Returning to the derivations of simple beam theory, the one issue remaining is to calculate the shear stresses in the beam. We would like to obtain an expression for $\sigma_{zx}(z)$.

Recall:

Shear stresses linked to axial (bending) stresses via:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \Rightarrow \frac{\partial \sigma_{zx}}{\partial z} = -\frac{\partial \sigma_{xx}}{\partial x} \tag{5}$$

and

Shear force (for case of rectangular cross-section, width b) linked to shear stress via:

$$S = -\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{xz} b \, dz \tag{7}$$

Multiply both sides of (5) by b and integrate from z to h/2 (we want to know $\sigma_{zx}(z)$

and we know that $\sigma_{zx}\left(\pm\frac{h}{2}\right) = 0$)
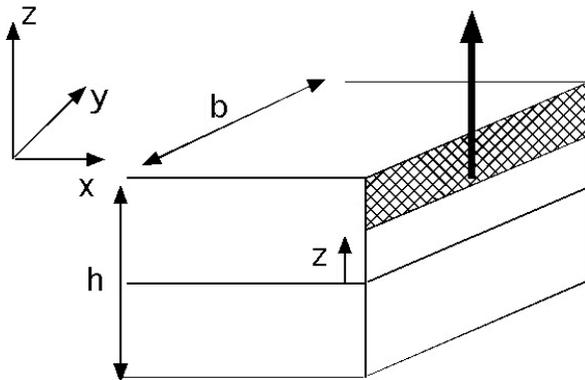
$$\int_z^{\frac{h}{2}} b\,\frac{\partial \sigma_{zx}}{\partial z}\,dz = -\int_z^{\frac{h}{2}} \frac{\partial \sigma_{xx}}{\partial x}\,b\,dz \qquad\qquad \sigma_{xx} = \frac{-Mz}{I_{yy}}$$

$$\left[b\sigma_{xz}(z) \right]_z^{\frac{h}{2}} = -\int_z^{\frac{h}{2}} \left(\frac{-dM}{dx} \right) \frac{zb}{I}\,dz$$

recall $\qquad \dfrac{dM}{dx} = S$

$$b\left(\sigma_{xz}\left(\frac{h}{2} \right) - \sigma_{xz}(z) \right) = \int_z^{h/2} S\left(\frac{zb}{I_{yy}} \right) dz$$

Note that at $z = \dfrac{h}{2}$ (top surface) $\sigma_{zx} = 0$

Also define

$$Q = \int_z^{h/2} zb\,dz \qquad\qquad \text{first moment of area above the center}$$

Hence:

$$\sigma_{xz}(z) = -\frac{SQ}{I_{yy}b} \qquad\qquad \text{shear stress -force relation}$$

For a rectangular section:
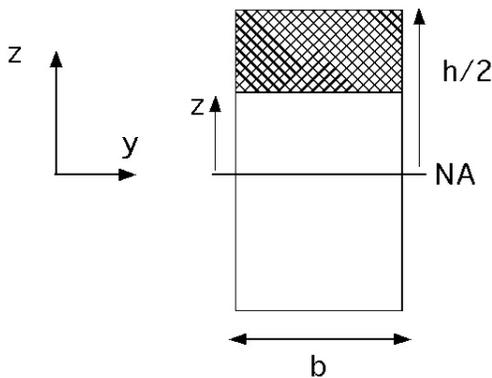
$$Q = \int_z^{h/2} zb\,dz = \left[\frac{z^2}{2}b \right]_z^{h/2} = \frac{b}{2}\left[\frac{h^2}{4} - z^2 \right]$$

Parabolic, maximum at $z = 0$. The centerline. Minimum = 0 at top and bottom "ligaments"
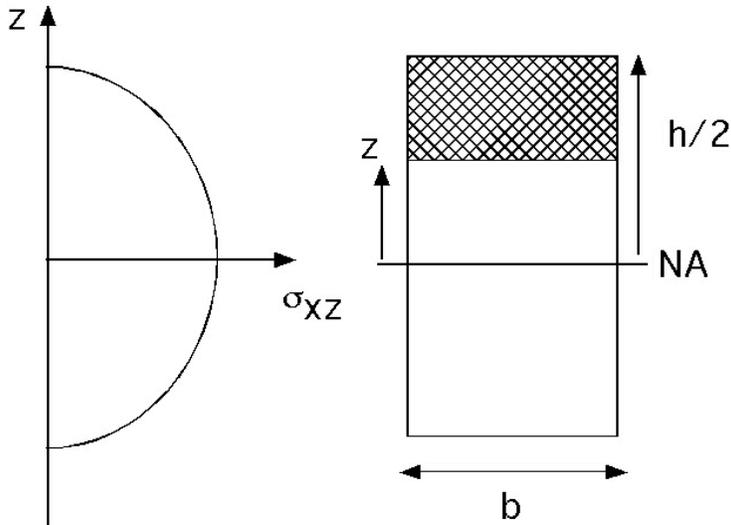
For non-rectangular sections this can be extended further if b is allowed to vary as a function of z. I.e. can write: $\sigma_{xz}(z) = -\dfrac{SQ}{I_{yy}b(z)}$
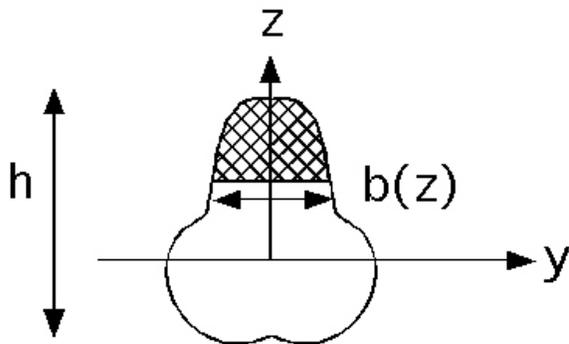
But need to beware cases in which we have thin walled open or closed sections where shear is not confined to σ_{xz}. Will also have σ_{xy} (on flanges). E.g.
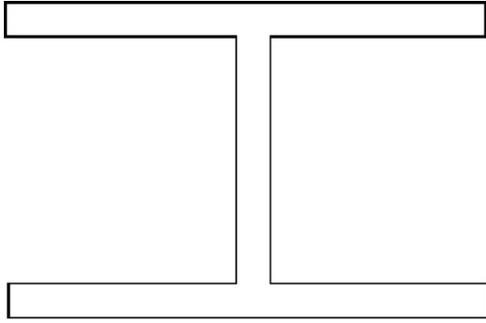
 or
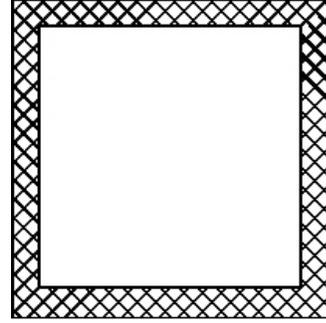
We will defer discussion of these cases until 16.20.

So this is the model for a simple beam. What can we do with it?

M7 Examples of Application of Beam Theory

Reading: Crandall, Dahl and Lardner 8.1.-8.3

<u>Example:</u> Calculate deflected shape and distribution of stresses in beam. Refer back to example problem in lecture M3.

Example: Uniform distributed load, q (per unit length, applied to simply supported beam.

Free Body diagram:

qL/2   qL/2
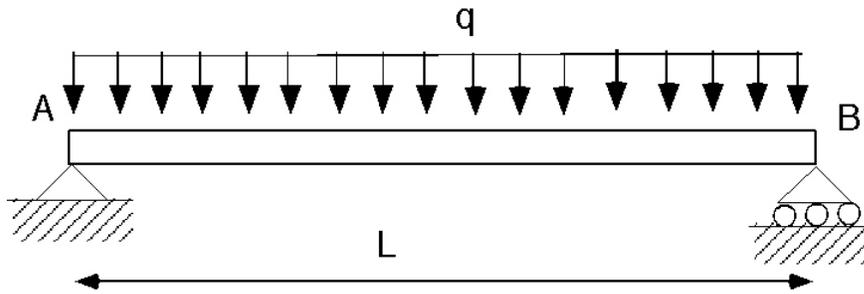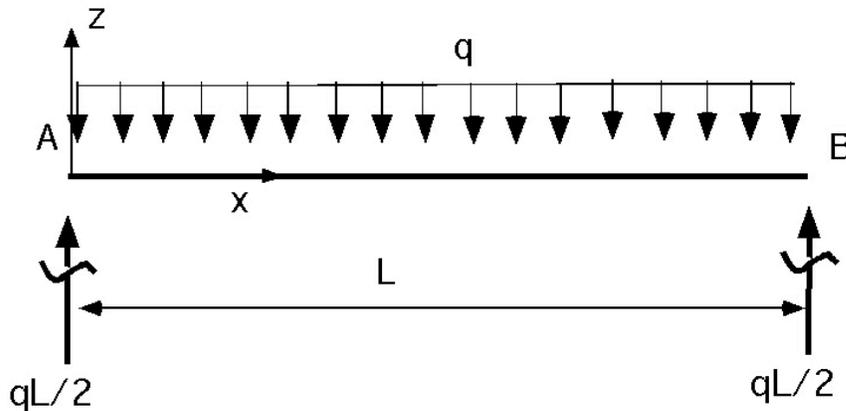
Apply method of sections to obtain bending moments and shear forces:

$$S(x) = \frac{qL}{2} - qx:$$
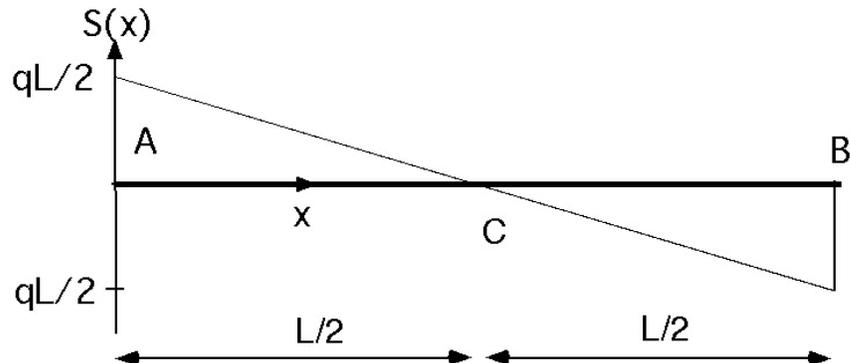
$$M(x) = \frac{qLx}{2} - \frac{qx^2}{2} :$$
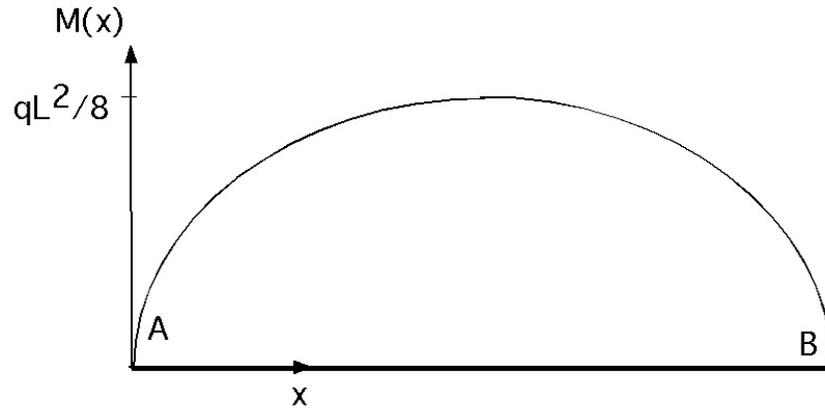
Let beam have rectangular cross-section, h thick, b wide. Material, Young's modulus E:
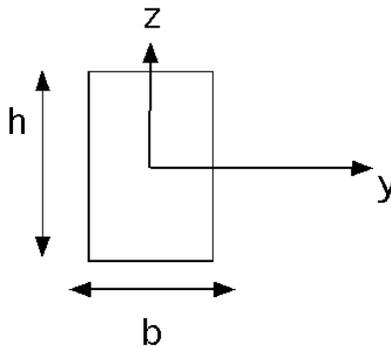
Deflected shape of the beam: obtain via moment curvature relationship:

$$EI \frac{d^2 w}{dx^2} = M \Rightarrow \frac{d^2 w}{dx^2} = \frac{M(x)}{EI}$$

Integrate to obtain deflected shape w(x)

$$w(x) = \int \int \frac{M(x)}{EI} dx\, dx$$

and for EI constant over length of the beam:

$$\Rightarrow w(x) = \frac{1}{EI} \int \int M(x)$$

For our case:

$$M(x) = \frac{-qx}{2}(x - L)$$

$$\frac{dw}{dx} = -\frac{1}{EI}\int \frac{qx}{2}(x - L)dx$$

$$= \frac{-q}{EI}\left(\left[\frac{x^3}{6} - \frac{Lx^2}{4}\right]\right) + C_1$$

$$So \quad w(x) = \frac{1}{EI}\int (\frac{q}{2}\left[\frac{x^3}{6} - \frac{Lx^2}{4}\right] + C_1)dx$$

$$w(x) = \frac{-q}{2EI}\left[\frac{x^4}{24} - \frac{Lx^3}{12} + C_1 x + C_2\right]$$

C_1 and C_2 are calculated from values of w(x) and w'(x) at boundary conditions.

Two constants of integration, need two boundary conditions to solve.

$$x = 0, \quad w = 0 \quad \Rightarrow C_2 = 0$$
$$x = L, \quad w = 0$$

$$0 = \frac{-L^4}{24} + \frac{L^4}{12} + C_1 L = 0 \Rightarrow C_1 = \frac{-L^3}{24}$$

Hence:

$$w(x) = -\frac{q}{48EI}\left[x^4 - 2Lx^3 + L^3 x\right]$$

Maximum deflection occurs where: $\dfrac{dw(x)}{dx} = 0$

$$\frac{dw}{dx} \Rightarrow 4x^3 - bLx^2 + L^3 = 0 \quad at \ x = \frac{L}{2}$$

$$w = -\frac{5}{384}\frac{qL^4}{EI_{yy}}$$

Find Stresses:

$$\sigma_{xx} = -\frac{Mz}{I_{yy}}$$ substitute for M(x)
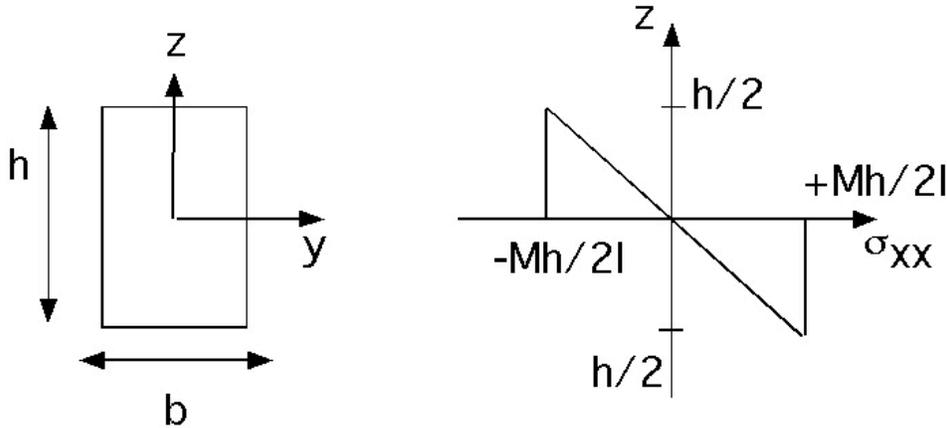
Max tensile (bending) stress at $x = \dfrac{L}{2}$, $M_{max} = \dfrac{qL^2}{8}$, $z = \pm\dfrac{h}{2}$

$$\sigma_{xx_{max}} = \frac{3qL^2}{4bh^2}$$

Shear stresses:

$$\sigma_{xz} = \frac{-SQ}{Ib}$$ occurs where max shear force $S = \dfrac{qL}{2}$ @ x=0, x=L

for rectangular cross-section

$$Q(z) = \int_{z}^{h/2} z'b\,dz'$$

$$= \left[\frac{bz'^2}{z'}\right]_{z}^{h/2} = \frac{b}{2}\left[\frac{h^2}{4} - z^2\right]$$

$$\sigma_{xz_{max}} = \frac{bh^2}{8}\frac{qL}{2}\frac{12}{bh^3}\frac{1}{b} = \frac{3}{4}\frac{qL}{bh}$$

Note that this is much smaller than $\sigma_{xx_{max}}$ if L/h is large (i.e slender beam).

M8 Application of Simple Beam Theory: Discontinuous Loading and Statically Indeterminate Structures

See Crandall, Dahl and Lardner 8.3, 3.6, 8.4

Example

P   P

A   B

L   L   L

Free body diagram

P   P

A   B

X

P   L   L   L   P

Shear forces:

S(x)

P

-P

X

Bending moments:

M(x)
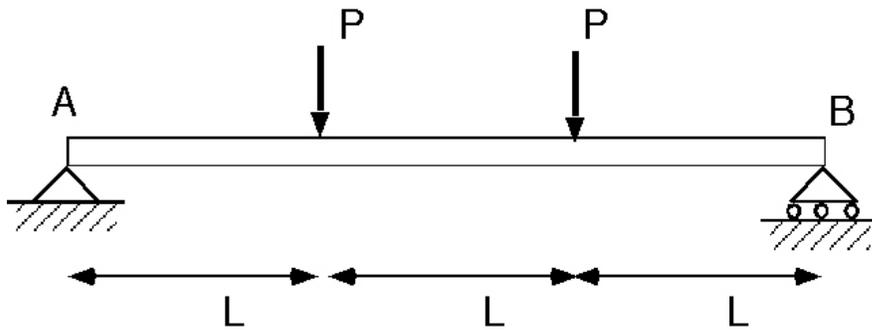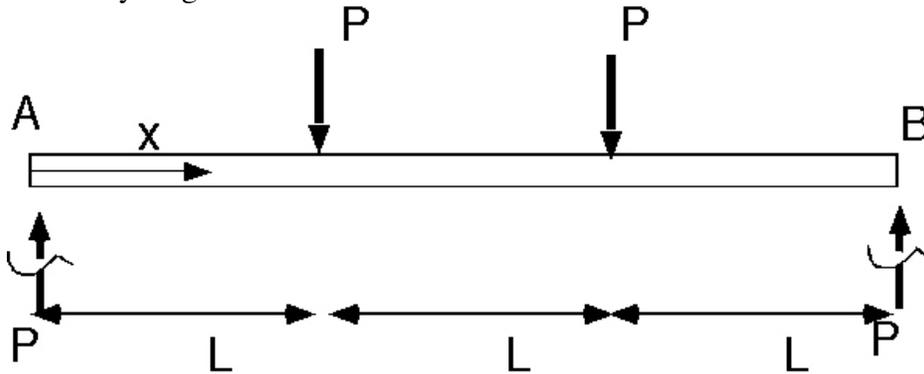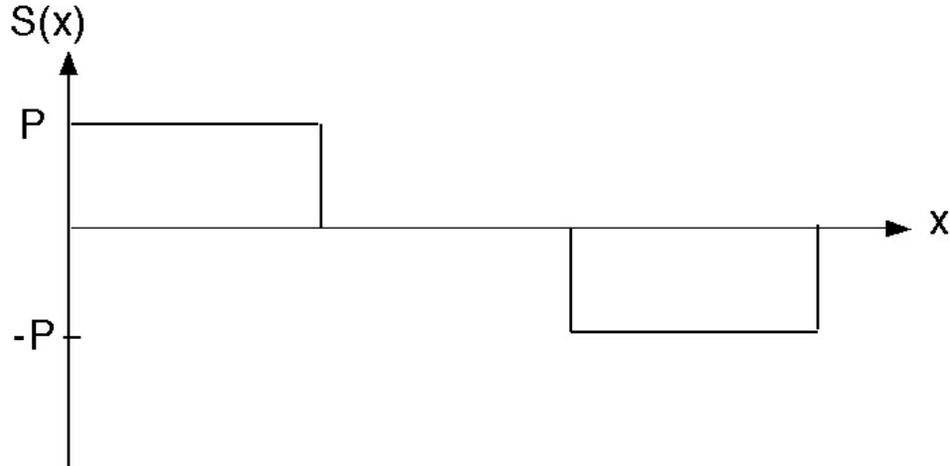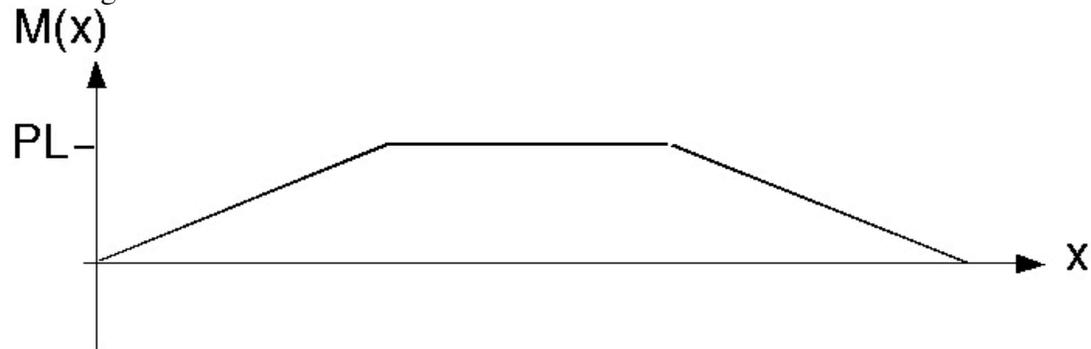
Note in passing M(x) and hence $\sigma_{xx} = \dfrac{-Mz}{I}$ = constant for $L \leq x \leq 2L$ - useful for mechanical testing of materials – "four point bending"

Could solve for deflections by writing separate equations for: M(x) for $0 < x < L$, $L < x < 2L$, $2L < x < 3L$ and integrating each equation separately and matching slope and displacements at interfaces between sections. This is a tedious process.

Or we can use:.

Macaulay's Method

Relies on the knowledge that the integral of the moment (i.e. the slope, and deflection) are continuous, smooth functions. Beams do not develop "kinks". This is a particular example of a class of mathematical functions known as "singularity functions" which are discontinuous functions, defined by their integrals being continuously defined.

Apply method of sections to the right of all the applied loads
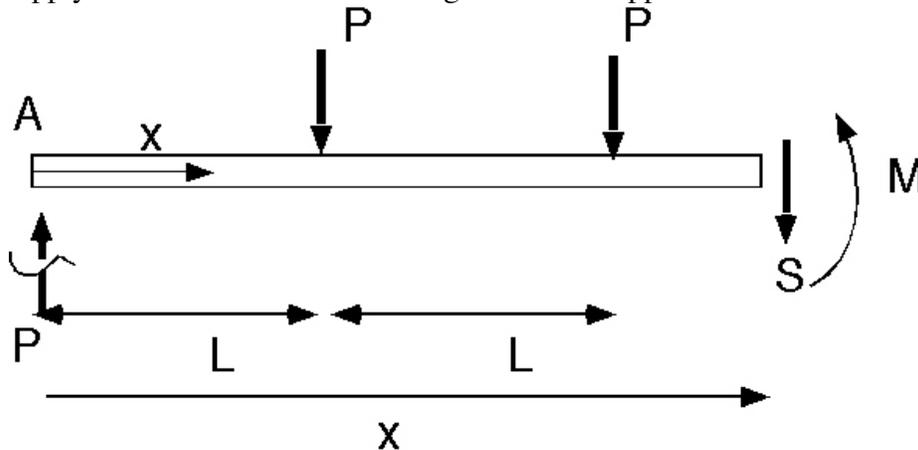
Write down equilibrium equation for section, leaving quantities such as {x-L} grouped within curly brackets { }

$$M_{x=x} = 0 \qquad\qquad M + P\{x - 2L\} + P\{x - L\} - Px = 0$$

{x-a} is treated as a single variable, only takes a value for $x > a$ otherwise { } = 0

Proceed as before - integrate moment-curvature relationship twice, applying boundary conditions

$$M = Px - P(x - 2L) - P(x - L) = EI\frac{d^2w}{dx^2}$$

$$EI\frac{dw}{dx} = \frac{Px^2}{2} - \frac{P}{2}\{x - 2L\}^2 - \frac{P}{2}\{x - L\}^2 + A$$

$$EIw = \frac{Px^3}{6} - \frac{P}{6}\{x - 2L\}^3 - \frac{P}{6}\{x - L\}^3 + Ax + B$$

apply boundary conditions: $w = 0$ @ $x = 0$

$$w = 0 \text{ @ } x = 3L$$

for $x = 0$ only $\dfrac{P(x)^3}{6}$ is evaluated = 0, therefore $B = 0$

for $x = 3L \Rightarrow \dfrac{PL^3}{6} - \dfrac{PL^3}{6} - \dfrac{P}{6}\left(8L^3\right) + A(3L) = 0$

$$\Rightarrow A = \frac{4}{9}PL^2$$

$$w(x) = \frac{1}{EI}\left[\frac{Px^3}{6} - \frac{P}{6}\{x - 2L\}^3 - \frac{P}{6}\{x - L\}^3 + \frac{4}{9}PL^2 x\right] \Leftarrow$$

Provides complete description of deflected shape of beam in one equation. Can also use for distributed loads.

Statically Indeterminate Beams

Reading: Crandall, Dahl and Lardner, 8.4

(Several ways to approach.)

1.) By simultaneous application of three great principles

i.e., Set up: equilibrium

Moment - curvature

Geometric Constraints

Solve simultaneously
(reactions remain as unknowns in moment-curvature equations)

2.) Superposition

Decompose problem into several statically determinate structures.

Replace constraints by applied loads

Solve for loads required to achieve geometric constraints.

Example: cantilever beam under uniform distributed load, with pin at right hand end.

A
q

B

$R_B = ?$

L

Equivalent to:

(1)

A
q

B

δ1

L

+

(2)

A

δ2

L

R_B
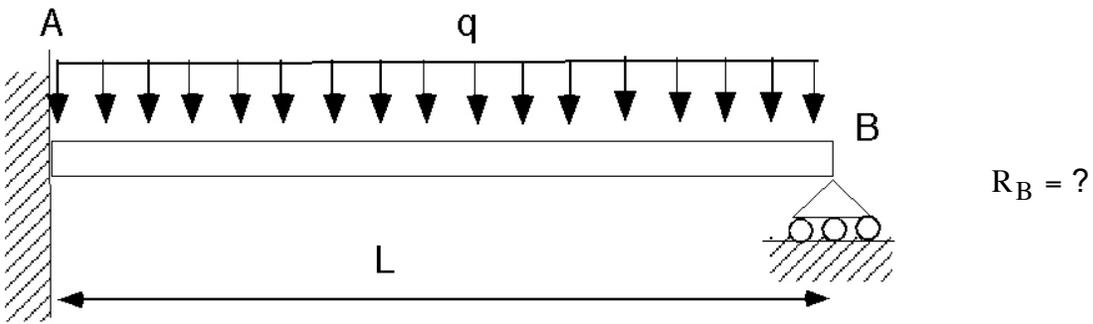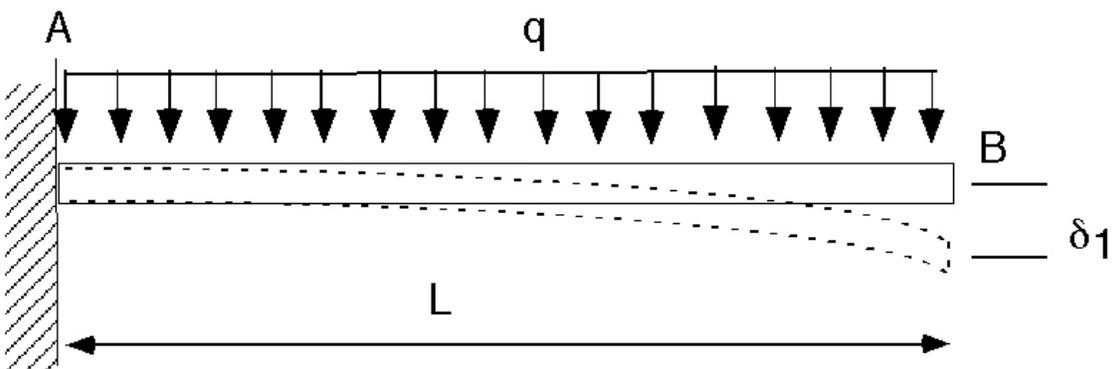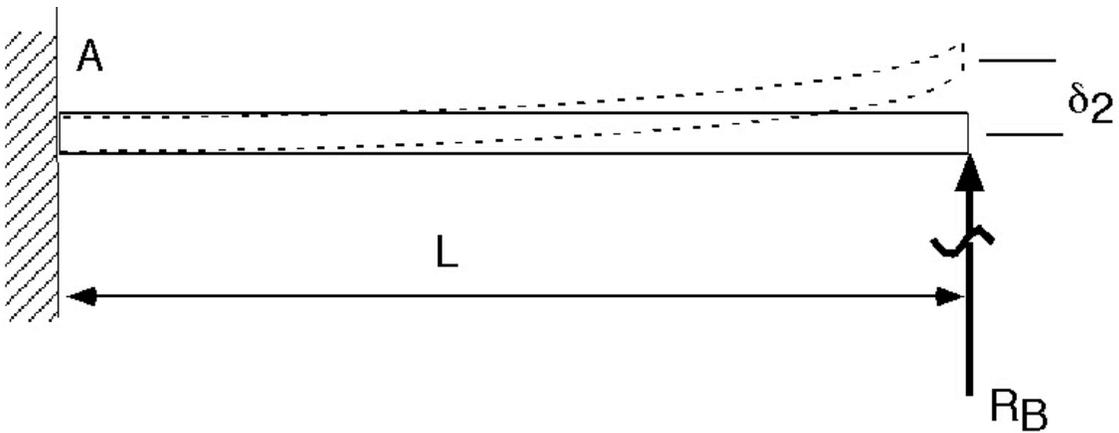
From "standard solutions" or calculation obtain expressions for deflections in case (1) and case (2):

$$\delta_1 = -\frac{q\ell^4}{8EI},$$

$$\delta_2 = +\frac{R_B L^3}{3EI}$$

$$\delta_1 + \delta_2 = 0$$

$$\Rightarrow \quad \frac{qL^4}{8EI} = \frac{R_B L^3}{3EI} \quad \Rightarrow \quad R_B = \frac{3qL}{8}$$

This is a powerful technique for solving beam problems – statically indeterminate, or otherwise complex loading.

Note. For superposition to work we must be superimposing the loads and boundary conditions on the same beam. Only change one variable in each loading case.